MA 2101 : Analysis I

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Exercise 1 Find two collections of nonempty sets $\{I_n\}_{n\in\mathbb{N}}$ and $\{J_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $I_{n+1} \subset I_n$ and $J_{n+1} \subset J_n$ for all $n \in \mathbb{N}$ such that

(i) Each set I_n is closed and $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$.

(ii) Each set J_n is bounded and $\bigcap_{n\in\mathbb{N}} J_n = \emptyset$.

Solution

(i) Let $I_n = [n, \infty)$ for all $n \in \mathbb{N}$. Note that all I_n are closed, since if there was a limit point $x \notin I_n$, then $x < n$. Setting $r = (n - x)/2$, we see that $B_r(x) \cap I_n = \emptyset$, which is a contradiction.

Let
$$\bigcap_{n\in\mathbb{N}} I_n = S,$$
and suppose $x \in S$. This means that $x \in I_n$ for all $n \in \mathbb{N}$, which requires $x > n$ for all $n \in \mathbb{N}$. This is absurd since the natural numbers are unbounded. Thus, $S = \emptyset$.

(ii) Let $J_n = (0, 1/n)$ for all $n \in \mathbb{N}$. Note that each J_n is an open interval in $\mathbb{R}$, or an open ball of radius $1/2n$ centred at $1/2n$, and is hence an open set in $\mathbb{R}$.

Let
$$\bigcap_{n\in\mathbb{N}} J_n = S,$$
and suppose $x \in S$. This means that $x \in J_n$ for all $n \in \mathbb{N}$, which requires $0 < x < 1/n$ for all $n \in \mathbb{N}$. This means that $n < 1/x$ for all $n \in \mathbb{N}$, which is again absurd since the natural numbers are unbounded. Thus, $S = \emptyset$.

Exercise 2 Let (M, d) be a metric space and let $A \subseteq M$. Prove that the boundary of A is given by the intersection of the closure of A with the closure of A^c.

Solution Let ∂A denote the boundary of A. We first show that $\partial A \subseteq \overline{A} \cap \overline{A^c}$. Pick $x \in \partial A$. By definition, $x \in \overline{A} \setminus A^0$. This means that there is no neighbourhood of x wholly contained within A. In other words, every neighbourhood of x contains points in A^c, so $x \in \overline{A^c}$. Thus, $x \in \overline{A} \cap \overline{A^c}$, so $\partial A \subseteq \overline{A} \cap \overline{A^c}$.

We now show that $\overline{A} \cap \overline{A^c} \subseteq \partial A$. Pick $x \in \overline{A} \cap \overline{A^c}$. This means that every neighbourhood of x contains points both from A and A^c, so it is not an interior point of A. Thus, $x \in \overline{A} \setminus A^0$, which means that $x \in \partial A$. Thus, $\overline{A} \cap \overline{A^c} \subseteq \partial A$.

Both inclusions show that $\partial A = \overline{A} \cap \overline{A^c}$, as desired.

Exercise 3 Show that the set $[0, \sqrt{2}) \cap \mathbb{Q}$ is a closed set in $\mathbb{Q}$ but not compact.

Solution We first show that $S = [0, \sqrt{2}) \cap \mathbb{Q}$ is closed in $\mathbb{Q}$. Suppose that $x \notin S$, $x \in \mathbb{Q}$ is a limit point of S. If $x < 0$, then set $r = -x/2 > 0$, so $B_r(x) \cap S = (3x/2, x/2) \cap [0, \sqrt{2}) \cap \mathbb{Q} = \emptyset$. If $x > \sqrt{2}$, then set $r = (x - \sqrt{2})/2$, so $B_r(x) \cap S = ((x + \sqrt{2})/2, (3x - \sqrt{2})/2) \cap [0, \sqrt{2}) \cap \mathbb{Q} = \emptyset$. In both cases, we reach a contradiction, which means that no limit points of S exist outside S in $\mathbb{Q}$. Thus, S is closed in $\mathbb{Q}$.

We now show that S is not compact. It is sufficient[1] to show that S is not compact in $\mathbb{R} \supset \mathbb{Q}$. Define $\mathcal{O}_n = (0, \sqrt{2} - \frac{1}{n})$ for all $n \in \mathbb{N}$, and $\mathcal{O}_0 = (-\frac{1}{4}, \frac{1}{4})$. Note that $0 \in \mathcal{O}_0$, and for every $x \in (0, \sqrt{2})$, $x \in \mathcal{O}_n$ for some n. This is true because for every $0 < x < \sqrt{2}$, we can write $x = \sqrt{2} - \epsilon$ for $\epsilon > 0$, so from the Archimedean property, we choose $n > 1/\epsilon$. Thus, $0 < x = \sqrt{2} - \epsilon < \sqrt{2} - \frac{1}{n}$, so $x \in \mathcal{O}_n$. Also, all $\mathcal{O}_n$ are open intervals in $\mathbb{R}$, hence open sets. Thus $\{\mathcal{O}_n\}_{n=0}^{\infty}$ is an open cover of $[0, \sqrt{2})$, hence an open cover of S.

Suppose $\{\mathcal{O}_n\}_{n=0}^{\infty}$ admits a finite subcover, $\{\mathcal{O}_n\}_{n \in J}$ for some finite indexing set of integers J. Set $m = \max J$. Clearly $m \neq 0$, since the set $\mathcal{O}_0$ does not cover S (note that $1 \notin \mathcal{O}_0$). From the density of the rational in the reals, we can choose a rational number p such that $\sqrt{2} - \frac{1}{m} < p < \sqrt{2}$. Thus, $p \in S$, but $p \notin \mathcal{O}_m$. Also, note that $\mathcal{O}_{n+1} \supset \mathcal{O}_n$ for all $n \in \mathbb{N}$, so $p \notin \mathcal{O}_n$ for any $n \in J$. In addition, $p > \sqrt{2} - 1 > \frac{1}{4}$, so $p \notin \mathcal{O}_0$. Thus, p is not contained in any finite subcover of $\{\mathcal{O}_n\}_{n=0}^{\infty}$, so S is not compact.

Exercise 4 Let A and B be compact subsets of a metric space (M, d). Show that both $A \cup B$ and $A \cap B$ are compact.

Solution We first show that $A \cup B$ is compact. Note that any open cover $\mathcal{C}$ of $A \cup B$ is also an open cover of A and an open cover of B, since $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Thus, $\mathcal{C}$ admits a finite subcover of A as well as a finite subcover of B. The union of these is a finite subcover of $A \cup B$, which proves that it is compact.

We now show that $A \cap B$ is compact. Note that A is closed in M, since it is compact. Since the intersection of a closed set and a compact set is compact, $A \cap B \subseteq A$ is compact.

[1] Recall that a set $S \subseteq T$ is compact in $T \subset M$ iff it is compact in M.