MA 2101 : Analysis I

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Exercise 1 Show that no interval in $\mathbb{R}$ is a union of two disjoint nonempty open sets.

Solution Let $I \subseteq \mathbb{R}$ be an interval such that $I = A \cup B$ where A and B are disjoint, nonempty open sets. Pick $a \in A$ and $b \in B$, and without loss of generality[1] let $a < b$. We construct the set

$$S = \{x : a \leq x \leq b, x \in A\} = [a, b] \cap A.$$

Note that $S \subseteq \mathbb{R}$ is bound above and below, and $a \in S$, so S has a supremum, say $\gamma = \sup S$. We have the restriction $a \leq \gamma \leq b$. Since $a, b \in I$, every element in between them must be in I since it is an interval. Thus, $\gamma \in I$. This means that γ must be in exactly one of A and B.

Suppose $\gamma \in A$. This means that $\gamma \neq b$, so $b - \gamma > 0$. Also, from the openness of A, we find $r > 0$ such that $(\gamma - r, \gamma + r) \subseteq A$. Setting $\epsilon = \min(r, b - \gamma)$, we see that $\gamma' = \gamma + \epsilon/2 \in A$, and $a \leq \gamma < \gamma' < b$, so $\gamma' \in S$. This is a contradicts the fact that $\gamma = \sup S$.

Similarly, suppose $\gamma \in B$. This means that $\gamma \neq a$, so $\gamma - a > 0$. Also, from the openness of B, we find $r > 0$ such that $(\gamma - r, \gamma + r) \subseteq B$. Setting $\epsilon = \min(r, \gamma - a)$, we see that $\gamma' = \gamma - \epsilon/2 \in B$, and $a < \gamma' < \gamma \leq b$, so $\gamma' \in S$. This means that there are no elements of A between γ' and γ, which means that γ' is also an upper bound of S. This again contradicts the fact that $\gamma = \sup S$ is the lowest upper bound.

Thus, we conclude that $\gamma \notin A$ and $\gamma \notin B$, so $\gamma \notin A \cup B = I$, which is absurd. Thus, it is impossible to choose such A and B, and this proves the desired statement.

Exercise 2 Given $n \in \mathbb{N}$, construct a bounded set $S_n \in \mathbb{R}$ which has exactly n limit points.

Solution Let $A_m = \{m + \frac{1}{n} : n > 1, n \in \mathbb{N}\}$. Clearly, A_m is bound by m and $m + 1$. We claim that

$$S_n = \bigcup_{m=1}^{n} A_m$$

has exactly n limit points.

Suppose $x \in \{1, \dots, n\}$. Then, for any neighbourhood $(x - \epsilon, x + \epsilon)$, we find $k \in \mathbb{N}$ such that $k\epsilon > 1$ using the Archimedean principle. Thus, $x - \epsilon < x + \frac{1}{k} < x + \epsilon$, so this neighbourhood of x contains the point $x + \frac{1}{k} \in A_x \subseteq S_n$, which means that x is a limit point of S_n.

Suppose $x \in S_n \setminus \{1, \dots, n\}$. We find $m, n \in \mathbb{N}$, $n > 1$ such that $x = m + 1/n$. Now,

$$\left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n(n-1)} = \left|\frac{1}{n-1} + \frac{1}{n}\right|.$$

Thus, setting $\epsilon = \frac{1}{n} - \frac{1}{n+1}$, we see that $(x - \epsilon, x + \epsilon) \cap S_n = \{x\}$, which means that x is not a limit point of S_n.

Suppose $x \notin S_n$. Note that the largest element of S_n is $n + \frac{1}{2}$, so if $x > n + \frac{1}{2}$, we set $\epsilon = (x - n - \frac{1}{2})/2$. Every element of S_n is greater than 1, so if $x < 1$, set $\epsilon = (1 - x)/2$. In both cases, we find that $(x - \epsilon, x + \epsilon) \cap S_n = \emptyset$. Otherwise, note that x cannot be an integer, since the only integers between 1 and $n + \frac{1}{2}$ are already in S_n. Thus, we find $m \in \mathbb{N}$ such that $m < x < m + 1$, i.e. $m = \lfloor x \rfloor$. Note that

[1] If $a > b$, just swap the roles of A and B. Note that $a \neq b$ since A and B share no common elements.

$m \in \{1, \dots n\}$. If $x \geq m + \frac{1}{2}$, set $\epsilon = (m + 1 - x)/3$, and note that $(x - \epsilon, x + \epsilon) \cap S_n = \emptyset$. Otherwise, $m < x < m + \frac{1}{2}$, so set $\epsilon = (x - m)/2$. Any points of S_n in the interval $(x - \epsilon, x + \epsilon)$ must be of the form $m + \frac{1}{k}$, where $m + \frac{1}{k} > x - \epsilon = (x + m)/2$, so $1/k > (x - m)/2$, so $k < 2/(x - m)$. Thus, there are finitely many such k, so $(x - \epsilon, x + \epsilon) \cap S_n$ is finite, which means that x is not a limit point of S_n.

This covers all possible cases of $x \in \mathbb{R}$, so we have proved that S_n has exactly n limit points.

Exercise 3 Let (M, d) be a metric space and A, B be two subsets of M. Show that $\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$. Is it always true that $\text{cl}(A \cap B) = \text{cl}(A) \cap \text{cl}(B)$?

Solution Given $A, B \subseteq M$, let $x \in \text{cl}(A \cap B)$. Thus, every neighbourhood of x contains at least one point of $A \cap B$. This point is in both A and B. This means that every neighbourhood of x contains a point in A, so x is a closure point of A. Similarly, every neighbourhood of x contains a point in B, so x is a closure point of B. Thus, $x \in \text{cl}(A)$ and $x \in \text{cl}(B)$, so $x \in \text{cl}(A) \cap \text{cl}(B)$. This proves that $\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$.

It is not always true that $\text{cl}(A \cap B) = \text{cl}(A) \cap \text{cl}(B)$. For example, let $A = (0, 1)$ and $B = (1, 2)$ be subsets of $\mathbb{R}$ with the usual topology. Then $\text{cl}(A) = [0, 1]$, $\text{cl}(B) = [1, 2]$ and $\text{cl}(A \cap B) = \text{cl}(\emptyset) = \emptyset$, while $\text{cl}(A) \cap \text{cl}(B) = \{1\}$.

Exercise 4 Let (M, d) be a metric space and A, B be two subsets of M. Show that $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.

Solution First, let $x \in \text{cl}(A)$. Then, x is a closure point of A, so every neighbourhood of x contains some point in A, which is also a point in $A \cup B$. Thus, x is a closure point of $A \cup B$, so $\text{cl}(A) \subseteq \text{cl}(A \cup B)$. A similar argument with B shows that $\text{cl}(B) \subseteq \text{cl}(A \cup B)$. Thus, $\text{cl}(A) \cup \text{cl}(B) \subseteq \text{cl}(A \cup B)$.

Now, let $x \in \text{cl}(A \cup B)$. Let $(A \cup B)'$ denote the set of limit points of $A \cup B$. We thus have $\text{cl}(A \cup B) = A \cup B \cup (A \cup B)'$. If $x \in A \cup B$, then we are done, since in that case, x is in one of A or B, which are included in their closures, so either $x \in \text{cl}(A)$ or $x \in \text{cl}(B)$.
Otherwise, $x \in (A \cup B)'$, i.e. x is a limit point of $A \cup B$. In this case, every neighbourhood of x contains infinitely many elements of $A \cup B$. Consider the neighbourhoods $N_{\epsilon_n}(x)$ of size $\epsilon_n = 1/n > 0$. In each of these neighbourhoods, we find a point $y_n \in A \cup B$. Suppose that $x \notin \text{cl}(A)$. This means that for some $k \in \mathbb{N}$, the ϵ_k neighbourhood of x contains no points in A. Thus, $y_k \in B$. However, since $N_{\epsilon_{n+1}} \subseteq N_{\epsilon_n}$ for all $n \in \mathbb{N}$, this forces $y_{k+1} \in B$. By induction, all $y_{n \geq k} \in B$, so $x \in \text{cl}(B)$. This follows because every neighbourhood larger than ϵ_k contains y_k, and for all smaller ϵ neighbourhoods, there exists $n \in \mathbb{N}$ such that $1/k = \epsilon_k \geq \epsilon > 1/n$, so $N_\epsilon \supseteq N_{\epsilon_n}$ which contains $y_n \in B$, since $n \geq k$. Thus, in all cases, $x \in \text{cl}(A) \cup \text{cl}(B)$, so $\text{cl}(A \cup B) \subseteq \text{cl}(A) \cup \text{cl}(B)$.

Hence, both inclusions prove that $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.