MA 2101 : Analysis I

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Exercise 1 Show that every nonnegative integer n has a decimal expansion of the form

$$n = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_0,$$

with $a_0, \dots, a_k \in \{0, \dots 9\}$.

Solution We first state and prove Euclid's Division Algorithm. Let a, b be nonnegative integers, $b \neq 0$. Then there exist unique integers q, r such that $a = bq + r$ and $0 \leq r < b$. To prove this, consider the set $S = \{a - qb : a - qb \geq 0, q \in \mathbb{Z}\}$. Now, $a = a - 0b \in S$, so S is a non-empty set of nonnegative integers. Thus, it has a minimal element (Well Ordering Principle), say $r = a - qb \geq 0$ for some $q \in \mathbb{Z}$. Thus, $a = bq + r$. Now, if $r \geq b$, then $r - b = a - (q+1)b \geq 0$, which contradicts the minimalilty of r in S. This forces $0 \leq r < b$.
We must now show that q, r are unique. Let $a = bq' + r'$ for some integers q', r' where $0 \leq r' < b$. Now, $b(q - q') + (r - r') = a - a = 0$, so $b(q - q') = r' - r$. Suppose $q - q' \neq 0$. Then, $|b(q - q')| > |b| = b$, but we already have $|r' - r| < b$, which is a contradiction. Hence, $q = q'$, so $r = r'$ and our solution is unique.

Note that if $a = 0$, then $q = r = 0$. Otherwise, $0 < a = bq + r < bq + b = b(q+1)$, so $q \geq 0$. Thus, it is possible to reiterate this process by using q as our new a.

With this, we supply an algorithm to obtain the coefficients a_i. Set $n = n_0$. If $n = 0$, then we are done, trivially ($a_0 = k = 0$). Otherwise, use Euclid's Division Algorithm to write $n_i = 10n_{i+1} + a_i$ and iterate over $i \in \{0, 1, 2, \dots, k\}$ while n_i is positive. This process must terminate, since $n_{i+1} = (n_i - a_i)/10 < n_i$, and the number of integers between 0 and n_0 is finite. Thus, we obtain the integers $\{a_i\}$ where $0 \leq a_i < 10$, and

$$n = a_0 + 10n_1 = a_0 + 10(a_1 + 10n_2) = \cdots = a_0 + 10(a_1 + 10(a_2 + \dots 10(a_{k-1} + 10n_k) \dots)).$$

We iterated while n_i was positive, so when we stopped, $n_k > 0$, and $n_{k+1} \leq 0$. We have already shown that the quotient $q \geq 0$ when $a > 0$, so $n_{k+1} \geq 0$. This forces $n_{k+1} = 0$, so $n_k = 10(0) + a_k = a_k$, where $0 < a_k < 10$. Thus, we distribute terms to obtain

$$n = a_0 + 10a_1 + 10^2 a_2 + \cdots + 10^k a_k,$$

as desired. Note that this also establishes the uniqueness of this representation, since any representation in base 10 demands $n = a_0 + 10(a_1 + 10a_2 + 10^2 a_3 + \cdots + 10^{k-1} a_k)$, where $0 \leq a_0 < 10$. Euclid's Division Algorithm guarantees the uniqueness of a_0 as well as the quotient $n_1 = a_1 + 10a_2 + \cdots + 10^{k-1} a_k$, upon which we recursively repeat the same argument.

Exercise 2 Let $A, B \subset \mathbb{Q}$ be two non-empty subsets such that every rational number is either in A or in B, and if $a \in A$ and $b \in B$, then $a < b$. Prove that there is a unique real number α such that every rational number less than α is in A and every rational number greater than α is in B.

Solution Note that A and B are subsets of $\mathbb{R}$. Since B is non-empty, we choose some $b \in B$ which is an upper bound for A since $a < b$ for all $a \in A$. Thus, A has a supremum. We claim that $\alpha = \sup A$ satisfies the desired properties, i.e. if $x < \alpha < y$ for some $x, y \in \mathbb{Q}$, then $x \in A$ and $y \in B$.

Let $x < \alpha$ for $x \in \mathbb{Q}$. This must be in exactly one of A and B. If it were in B, then for any $a \in A$, we have $a < x$. Thus, x is an upper bound of A. On the other hand, α is the least upper bound, so we must have $\alpha \leq x$, which is a contradiction. Thus, $x \in A$.

Let $\alpha < y$ for $y \in \mathbb{Q}$. Again, this must be in exactly one of A and B. If it were in A, that would force $y \le \alpha$ since α is an upper bound, which is a contradiction. Thus, $y \in B$.

It remains to show that α is unique. Suppose $\beta \in \mathbb{R}$ also satisfies the desired properties. If $\beta < \alpha$, then we find a rational number p such that $\beta < p < \alpha$ using the density of the rationals in the reals. This is a contradiction, since $p < \alpha$ implies that $p \in A$, but $\beta < p$ implies that $p \in B$. Similarly, if $\beta > \alpha$, we find a rational number q such that $\beta > q > \alpha$, which is a contradiction again, since $\beta > q$ implies that $q \in A$ but $q > \alpha$ implies that $q \in B$. Thus, the only possibility is $\alpha = \beta$. Hence, the real number satisfying the desired properties is unique.

Exercise 3 Let x be a positive real number and let

$$S_x = \{x, x^{1/2}, x^{1/3}, \dots, x^{1/n}, \dots\}.$$

Show that $\inf S_x = 1$ if $x \ge 1$ and $\sup S_x = 1$ if $x \le 1$.

Solution Note that $S_x \subset \mathbb{R}$. First, we take the case that $x > 1$. Note that $x^{1/n} > 1$ because

$$\Bigg(\underbrace{(x^{1/n})\dots(x^{1/n})}_{n \text{ times}}\Bigg)^n = \underbrace{(x^{1/n})^n\dots(x^{1/n})^n}_{n \text{ times}} = x^n$$

by commutativity of multiplication. From the uniqueness of positive n^{th} roots, we have

$$\underbrace{(x^{1/n})\dots(x^{1/n})}_{n \text{ times}} = x > 1,$$

which is only possible if $x^{1/n} > 1$. This means that S_x is bounded below, and thus has an infimum. We now show that $\inf S_x = 1$, i.e. given any $\epsilon > 0$, we must find some $s \in S_x$ such that $1 < s < 1 + \epsilon$. By the Archimedean property, we find $n \in \mathbb{N}$ such that $n\epsilon > x$, and we claim that $1 < x^{1/n} < 1 + \epsilon$. To prove this, set $x^{1/n} = 1 + h$, where $h > 0$. Using the binomial theorem,

$$x = (x^{1/n})^n = (1+h)^n = 1 + nh + \frac{1}{2}n(n-1)h^2 + \cdots > 1 + nh > nh.$$

Thus, $h < x/n < \epsilon$, so $1 < x^{1/n} < 1 + x/n < 1 + \epsilon$ as desired.

If $x < 1$, note that $1/x > 1$, so $\inf S_{1/x} = 1$. By a similar argument as before, we have $x^{1/n} < 1$ for all $n \in \mathbb{N}$, so S_x is bounded above and has a supremum. Also, note that $S_{1/x} = \{1/s : s \in S_x\}$ because $1/x^{1/n} = (1/x)^{1/n}$. Using the property proved in Assignment I, Exercise 5, we must have $(\sup S_x) \cdot (\inf S_{1/x}) = 1$, thus $\sup S_x = 1$.

In the special case that $x = 1$, note that the positive n^{th} root of 1 are all 1 as $1^n = 1$ for all $n \in \mathbb{N}$. Thus, $S_1 = \{1\}$, whose supremum and infimum are both trivially 1.

Exercise 4

(a) For $\alpha, \beta \in \mathbb{Q}$ with $\alpha < \beta$ and for $x \in \mathbb{R}_{>1}$, show that $x^\alpha < x^\beta$.

Solution Let $\alpha = m/q$ and $\beta = n/q$, for integers m, n, q, where $q > 0$ is chosen as a common denominator. Note that we must have $m < n$. Also note that for any $y > 1$, we must have $y^{1/q} > 1$, because

$$(y^{1/p})^p = \underbrace{(y^{1/q})\dots(y^{1/q})}_{q \text{ times}} = y > 1.$$

Note that by definition, $y^{1/p}$ is the unique positive real r such that $r^p = y$. Now, since $n - m > 0$, we have

$$x^{n-m} = \underbrace{(x)\dots(x)}_{n-m \text{ times}} > 1.$$

Hence, $x^{(n-m)/q} > 1$. Now, using the definition $x^{a/b} = (x^a)^{1/b}$, we have

$$x^{(n-m)/q} = (x^{n-m})^{1/q} = (x^n x^{-m})^{1/q} = x^{n/q} x^{-m/q} = x^\beta x^{-\alpha} > 1.$$

Since, $x^{-\alpha} = 1/x^\alpha$, we multiply x^α on both sides, obtaining $x^\beta > x^\alpha$, as desired.

(b) For $a, b \in \mathbb{R}$ with $a > 0$, let

$$a^b := \sup\{a^t : t \in \mathbb{Q}, t < b\}.$$

For $x, y \in \mathbb{R}$ with $x > 1$ and $y > 0$, let

$$\log_x y := \sup\{s : s \in \mathbb{Q}, x^s < y\}.$$

Show that

$$x^{\log_x y} = y.$$

Solution Let $x, y \in \mathbb{R}$ be fixed. We must show that $\sup S = y$, where

$$S = \{x^t : t \in \mathbb{Q}, t < \sup\{s : s \in \mathbb{Q}, x^s < y\}\}.$$

Also set

$$T = \{s : s \in \mathbb{Q}, x^s < y\}.$$

Pick an arbitrary element from S, say x^t for some $t \in \mathbb{Q}$. Note that $t < \sup T$, so there exists an element of T, say some $s \in \mathbb{Q}$, such that $t < s \leq \sup T$. Thus, from our previous exercise, we have $x^t < x^s$. Also, since $s \in T$, we have $x^s < y$. Thus, $x^t < y$ for all elements $x^t \in S$, so y is indeed an upper bound of S. This means that S has a supremum $\sup S \leq y$. Note that since $x > 1$, any $x^t > 0$, so $\sup S > 0$.

Suppose that $0 < \sup S = y' < y$. Then, $1 < y/y'$, so from Exercise 3, we find $n \in \mathbb{N}$ such that

$$1 < x^{1/n} < \frac{y}{y'},$$

i.e. $x^{1/n} y' < y$. Now, we pick $t \in \mathbb{Q}$ such that

$$\sup T - \frac{1}{n} < t \leq \sup T.$$

Note that this means that $t \in T$, and that $\sup T < t + 1/n \notin T$. Thus, $x^t \in S$, so $x^t \leq \sup S = y'$. Thus, $x^{t+1/n} \leq x^{1/n} y' < y$. However, this means that $t + 1/n \in T$, which is a contradiction. Thus, there exist no such y', so $\sup S \geq y \geq \sup S$, proving that $\sup S = y$.