MA 2101 : Analysis I

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Exercise 1 Show that for every real number r, there exists an integer such that $n \leq r < n + 1$.

Solution Supoose that there is no $n \in \mathbb{Z}$ such that $n \leq r < n + 1$ for some $r \in \mathbb{R}$. Note that the integers are unbounded below, so there exists some $m \in \mathbb{Z}$ such that $m \leq r$. By our assumption, we cannot have $r < m + 1$, so instead $m + 1 \leq r$. Let this be our base case.
Now, let $k \in \mathbb{Z}$ be such that $m \leq k \leq r$. Again, $r < k + 1$ would contradict our assumption, so $k + 1 \leq r$. Thus, we have shown by induction that all integers $n \geq m$ are bounded above by r. Additionally, the other integers $n' < m < r$ anyways. Thus, $n \leq r$ for all $n \in \mathbb{Z}$, which is absurd since the integers are unbounded above. This proves the given statement.

Exercise 2 Show that between any two rational numbers, there exists an irrational number.

Solution Without loss of generality, let $p, q \in \mathbb{Q}$ such that $p > q$. Note that $2 > \sqrt{2} > 1$, so $0 < 1/\sqrt{2} < 1$ and $0 < (p - q)/\sqrt{2} < p - q$. Adding q to both sides,

$$q < q + \frac{p - q}{\sqrt{2}} < p.$$

Note that the irrationality of $q + (p - q)/\sqrt{2}$ follows directly from the irrationality of $\sqrt{2}$.

Exercise 3 Show that in a group, every element has a unique inverse.

Solution Let $(G, *)$ be a group with identity $e \in G$, and let $a \in G$ be arbitrary. Clearly, a must have an inverse in G. Suppose $a', a'' \in G$ are two such inverses. Thus,

$$a' * a = e = a * a', \quad \text{and} \quad a'' * a = e = a * a''.$$

Now, we evaluate

$$\begin{aligned} a' &= a' * e && \text{(Identity)} \\ &= a' * (a * a'') && \text{(Composition with inverse)} \\ &= (a' * a) * a'' && \text{(Associativity)} \\ &= e * a'' && \text{(Composition with inverse)} \\ &= a'' && \text{(Identity)} \end{aligned}$$

Thus, $a' = a''$ for all inverses of a. In other words, the inverse of a is unique.

Exercise 4 Let $T \subset \mathbb{R}$ be bounded and let $S = \{|x - y| : x, y \in T\}$. Show that $\sup S = \sup T - \inf T$.

Solution We assume that T is non-empty. Note that T is a bounded subset of $\mathbb{R}$, so $\sup T$ and $\inf T$ exist by the completeness of $\mathbb{R}$. Without loss of generality, let $x, y \in T$ such that $x \geq y$. Then, $|x - y| \leq x - y \leq \sup T - \inf T$, since $x \leq \sup T$ and $y \geq \inf T$[1]. Hence, S is a subset of $\mathbb{R}$ bounded above, so $\sup S$ exists. We claim that $\sup S = \sup T - \inf T$. Thus, for any $\epsilon > 0$, we must find $s \in S$ such that $\sup T - \inf T - \epsilon < s \leq \sup T - \inf T$.
Now, from the properties of the supremum and infinum, we choose $x', y' \in T$ such that $\sup T - \epsilon/2 < x' \leq \sup T$ and $\inf T \leq y' < \inf T + \epsilon/2$. Thus, $x' - y' > \sup T - \inf T - \epsilon$. Thus, without loss of generality[2], we have $s = |x' - y'| \in S$ and $\sup T - \inf T - \epsilon < s \leq \sup T - \inf T$. Thus, $\sup T - \inf T$ is indeed the least upper bound of S, and is thus equal to its supremum.

[1] The analogous case with $x < y$ shows that $|x - y| = -x + y \leq -\inf T + \sup T$.

[2] If $y' > x'$, we can simply swap the roles of x' and y', since $\sup T - \epsilon/2 < x' < y' \leq \sup T$ and $\inf T \leq x' < y' < \inf T + \epsilon/2$.

Exercise 5 Find the supremum and infimum of the set $S = \{m/(m+n) : m, n \in \mathbb{N}\}$.

Solution We claim that $\inf S = 0$ and $\sup S = 1$. First, note that

$$0 \; < \; \frac{m}{m+n} \; < \; \frac{m}{m} = 1,$$

for all $m, n \in \mathbb{N}$. Thus, S is bounded, so its supremum and infimum exist by the completeness of $\mathbb{R}$. Also, we must have $\sup S \leq 1$ and $\inf S \geq 0$. We must now show that for any upper bound $1 > \alpha \in \mathbb{R}$ and for any lower bound $0 < \beta \in \mathbb{R}$ of S, there exist $x, y \in S$ such that

$$0 \; < \; x \; < \; \beta, \qquad \text{and} \qquad \alpha \; < \; y \; < \; 1.$$

Clearly, $1/2 = 1/(1+1) \in S$, so $\beta < 1/2 < 1$ and $\alpha > 1/2 > 0$.
On the other hand, the rationals $\mathbb{Q}$ are dense in the reals, so between any two real numbers, there exists a rational number p/q for $p, q \in \mathbb{Z}$, $q \neq 0$. Thus, we find rationals $0 < a/b < \beta < 1$ and $0 < \alpha < c/d < 1$, so $0 < a < b$ and $0 < c < d$ for $a, b, c, d \in \mathbb{N}$. Thus, $0 < b - a \in \mathbb{N}$ and $0 < d - c \in \mathbb{N}$. We thus set

$$x = \frac{a}{a + (b-a)} \in S, \qquad y = \frac{c}{c + (d-c)} \in S,$$

completing the proof.