MA 2101 : Analysis I

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Exercise 1 Show that $\sqrt{2}+\sqrt{3}$ is not rational.

Solution Assume to the contrary that $\sqrt{2}+\sqrt{3}$ is rational. We write $\sqrt{2}+\sqrt{3} = p/q$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. Then $(\sqrt{2}+\sqrt{3})^2 = 5 + 2\sqrt{6} = p^2/q^2$ is also rational, and so is $\sqrt{6} = (p^2 - 5q^2)/2q^2$.

Let $\sqrt{6} = a/b$ where $a, b \in \mathbb{Z}$, $b \neq 0$, and $\gcd(a, b) = 1$. Squaring and rearranging, we have $a^2 = 6b^2$. Since $6b^2$ is even, so is a^2, and so is a (this follows since 2 is a prime). Thus, we write $a = 2c$ for some integer c, hence $4c^2 = 6b^2 \implies 2c^2 = 3b^2$. Now, $2c^2$ is even, so $3b^2$ must be even as well. However, we already know that a is even and shares no common factors with b. Thus, b must be odd, and so is $3b^2$. This is a contradiction. Thus, $\sqrt{6}$ cannot be rational, so $\sqrt{2}+\sqrt{3} \notin \mathbb{Q}$.

Exercise 2 Let a be a real number such that $a > 1$ and let $S = \{a^n : n \in \mathbb{N}\}$. Show that the set S has no upper bound.

Solution Since $a > 1$, we write $a = 1 + x$ for some positive real x, then expand $(1 + x)^n$ using the binomial theorem to obtain the inequality

$$a^n \;=\; 1 + nx + \frac{1}{2}n(n-1)x + \cdots + x^n \;>\; nx.$$

Now, suppose that S is bounded above by some real number β. Clearly, $\beta > 1$ since $a^n > 1$. This would imply that $\beta > a^n > nx$ for all $n \in \mathbb{N}$. Thus, $n < \beta/x$ for all $n \in \mathbb{N}$, which is absurd, since $\mathbb{N}$ is unbounded in $\mathbb{R}$. Thus, S has no upper bound in the reals.

Exercise 3 Show that $\mathbb{N}$, the set of natural numbers, has the LUB property.

Solution Let $\emptyset \neq E \subseteq \mathbb{N}$. be bounded above. The Well-Ordering Principle tells us that E is bounded below as well, so the set E is finite. We show that E has a supremum, and that it is contained within E, by induction on the cardinality of E. As a base case, suppose E has exactly one element, so $E = \{x_0\}$. We claim that $\sup E$ exists and that $\sup E = x_0 = \max E$. This is clearly true since $x \leq x_0$ for all $x \in E$, and if $y \in \mathbb{N}$ is an upper bound of E, $x \leq y$ for all $x \in E$, so $x_0 \leq y$ in particular. Hence, x_0 is the maximum of the singleton E.

We now assume that $\max E$ exists for all finite subsets of $\mathbb{N}$ bound above containing exactly k elements. Let $\emptyset \neq D \subseteq \mathbb{N}$ containing exactly $k + 1$ elements be arbitrary. We choose and fix an arbitrary $d \in D$, then set $D' = D \setminus \{d\}$. Clearly, D' contains exactly k elements, so $d' = \max D'$ exists.
Now, if $d > d'$, then $d > d' \geq x'$ for all $x' \in D'$, so $d \geq x$ for all $x \in D$. Also, if $y \in \mathbb{N}$ is an upper bound of D, then $x \leq y$ for all $x \in D$, so $d \leq y$ in particular. Thus, $d = \sup D = \max D$.
Otherwise, if $d \leq d'$, then $d' \geq x$ for all $x \in D$. Again, if $y \in \mathbb{N}$ is an upper bound of D, then $x \leq y$ for all $x \in D$, so $d' \leq y$ in particular. Thus, $d' = \sup D = \max D$. Hence, every subset of $\mathbb{N}$ containing $k + 1$ elements and bound above has a maximum.

Therefore, by induction on k, all non-empty subsets of $\mathbb{N}$ bound above have a supremum. Thus, the set $\mathbb{N}$ has the LUB property.

Exercise 4 We know that if we input any positive natural number in a calculator and keep on pressing the square root button, finally we get 1. Show that if you do this experiment on an n-digit calculator, then starting with some positive number, the number of times you need to press the square root button to reach 1 is at most

$$1 + \left\lfloor \log_2(n+1) - \log_2 \log_{10}\left(1 + \frac{1}{10^{n-1}}\right) \right\rfloor.$$

Solution NOTE: We assume that the calculator displays the first **truncated** n digits of the **true value**. We do not take into account rounding errors introduced between steps. On the other hand, such errors can only truncate/round down the intermediate numbers, so our result still serves as an upper bound on the required number of steps.

Let the number initially entered be $x > 1$. After m presses of the square root button, we obtain the number $x^{1/2^m}$. Now, our calculator displays only n digits, so the number

$$\underbrace{1.000\ldots0}_{n \text{ digits}}abc\ldots$$

is displayed as simply $1.000\ldots0$ when truncated. Note that this number is at most $1 + 1/10^{n-1} := L$. Suppose our calculator finally displays $1.000\ldots0$, whereas the true answer is some $y < L$. Now, the calculator must have displayed some number not equal to 1 on the previous step, so $y^2 \geq L$. We now traceback the process of taking square roots by squaring y m times, to obtain the initial number $y^{2^m} \geq L^{2^m/2}$. Now, since our calculator only holds n digits, this initial number can be at most

$$\underbrace{999\ldots9}_{n \text{ digits}},$$

which is simply $10^n - 1$. Thus, we demand $y^{2^m} \leq 10^n - 1$, or

$$\begin{aligned}
L^{2^m/2} &< 10^n \\
2^{m-1}\log_{10} L &< n, \\
m - 1 + \log_2 \log_{10} L &< \log_2 n, \\
m - 1 &< \log_2 n - \log_2 \log_{10} L, \\
m - 1 &< 1 + \lfloor \log_2 n - \log_2 \log_{10} L \rfloor, \\
m &\leq 1 + \lfloor \log_2 n - \log_2 \log_{10} L \rfloor,
\end{aligned}$$

as desired. Here, have used the inequalities $x < 1 + \lfloor x \rfloor$ for $x \in \mathbb{R}$, and $p - 1 < q \implies p \leq q$ for $p, q \in \mathbb{Z}$.

Note that under our assumptions, if we start with $x < 1$, the result $x^{1/2^m}$ will always be of the form $0.abc\ldots$, which when truncated is never of the form equal to $1.000\ldots$.
If, however, we allow the number

$$\underbrace{0.999\ldots9}_{n \text{ digits}}5abc\ldots$$

to be rounded up to 1, then we proceed with a similar argument as above. Note that this number is at least $1 - 1/10^{n-1} + 5/10^n = 1 - 5/10^n := M$. Our final result must be some $w > M$, such that $w^2 \leq M$. Our initial value w^{2^m} must have been at least

$$\underbrace{0.000\ldots1}_{n \text{ digits}},$$

which is simply $1/10^{n-1}$. Thus, we demand $w^{2^m} \geq 10^{-n+1}$, or

$$\begin{aligned}
M^{2^m/2} &\geq 10^{-n+1} \\
2^{m-1}\log_{10} M &\geq -n + 1, \\
2^{m-1}\log_{10}(1/M) &\leq n - 1 < n, \\
m - 1 + \log_2 \log_{10}(1/M) &< \log_2 n, \\
m - 1 &< \log_2 n - \log_2 \log_{10}(1/M), \\
m &\leq 1 + \lfloor \log_2 n - \log_2 \log_{10}(1/M) \rfloor.
\end{aligned}$$

Note that for all $0 < \epsilon < 1/2$, we have $1 < 1/(1-\epsilon) < 1 + 2\epsilon$. This is equivalent to $(1-\epsilon)(1+2\epsilon) = 1 + \epsilon - 2\epsilon^2 = 1 + \epsilon(1 - 2\epsilon) > 1$, which is clearly true since $\epsilon < 1/2$. Thus, $1 < 1/M = 1/(1 - 5/10^n) < 1 + 2 \cdot 5/10^n = 1 + 1/10^{n-1} = L$, so $\log_{10}(1/M) < \log_{10} L$. Thus, the bound on m we obtain for $x < 1$ is weaker than the one for $x > 1$. Hence, our first result holds.

Exercise 5 Let S be a non-empty subset of the reals such that each element of S is greater than or equal to 1, and let $T = \{1/s : s \in S\}$. Then show that

(a) $\sup T \leq \inf S$.

(b) $(\sup T) \cdot (\inf S) = 1$.

Solution Note that $s \geq 1$, so $1/s \leq 1$ for all $s \in S$. Thus, S and T are non-empty subsets of $\mathbb{R}$, where S is bound below by 1 and T is bound above by 1, so $\sup T$ and $\inf S$ exist in $\mathbb{R}$.

(a) First, note that $\inf S \geq 1$. This is true because 1 is a lower bound of S, so by definition, $\inf S \geq l$ for all lower bounds l of S. Similarly, we also conclude that $\sup T \leq 1$, since 1 is an upper bound of T so $\sup T \leq u$ for all upper bounds u of T. Putting these together, $\sup T \leq 1 \leq \inf S$.

(b) First, note that $\inf S \leq s$ for all $s \in S$, so $1/\inf S \geq 1/s$ for all $s \in S$. Now, for each $t \in T$, there exists $s' \in S$ such that $t = 1/s'$, therefore $1/\inf S \geq t$ for all $t \in T$. In other words, $1/\inf S$ is an upper bound of T, so $1/\inf S \geq \sup T$, since $\sup T$ is the least upper bound. Similarly, note that $\sup T \geq t$ for all $t \in T$, so $1/\sup T \leq 1/t$ for all $t \in S$. Again, for each $s \in S$, there exists t' in T such that $s = 1/t'$, so $1/\sup T \leq s$ for all $s \in S$. Thus, S is bound below by $1/\sup T$, so $1/\sup T \leq \inf S$.

These two inequalities read $\sup T \cdot \inf S \leq 1 \leq \sup T \cdot \inf S$. Thus, by trichotomy, we must have $\sup T \cdot \inf S = 1$ as desired.