MA 1202 : Mathematical Methods I

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Problem 1 Check whether the function f, defined by

$$f(x,y) \;=\; \frac{x-1-iy}{(x-1)^2+y^2},$$

is analytic by the following methods.

(i) Using the Cauchy-Riemann equations.

(ii) Expressing f in the form $f(x,y) \equiv g(z,\bar{z})$.

Here, $x = \Re(z)$ *and* $y = \Im(z)$ *for* $z \in \mathbb{C}$.

Solution Note that using the identity $a^2+b^2 = (a+ib)(a-ib)$, we have

$$f(x,y) \;=\; \frac{1}{x-1+iy},$$

which is not defined at $z = 1$, i.e. $(x,y) = (1,0)$.

(i) We write

$$f(x,y) \;=\; u(x,y) \;+\; iv(x,y),$$

where

$$u(x,y) \;=\; \frac{x-1}{(x-1)^2+y^2}, \qquad \text{and} \qquad v(x,y) \;=\; \frac{-y}{(x-1)^2+y^2}.$$

Note that both u and v are continuous except at $z = 1$, where they are both undefined. Otherwise, on $z \in \mathbb{C} \setminus \{1\}$, we demand

$$\frac{\partial u}{\partial x} \;=\; \frac{\partial v}{\partial y}, \qquad \text{and} \qquad \frac{\partial u}{\partial y} \;=\; -\frac{\partial v}{\partial x}.$$

Indeed, notating $h_X \equiv \partial h/\partial X$, we have

$$u_x \;=\; \frac{[(x-1)^2+y^2]-[2(x-1)(x-1)]}{[(x-1)^2+y^2]^2} \;=\; \frac{-(x-1)^2+y^2}{[(x-1)^2+y^2]^2}$$

$$v_y \;=\; \frac{-[(x-1)^2+y^2]-[-y(2y)]}{[(x-1)^2+y^2]^2} \;=\; \frac{-(x-1)^2+y^2}{[(x-1)^2+y^2]^2}$$

$$u_y \;=\; \frac{0-[(x-1)(2y)]}{[(x-1)^2+y^2]^2} \;=\; \frac{-2(x-1)y}{[(x-1)^2+y^2]^2}$$

$$v_x \;=\; \frac{0-[-2y(x-1)]}{[(x-1)^2+y^2]^2} \;=\; \frac{2(x-1)y}{[(x-1)^2+y^2]^2}$$

Thus, $u_x = v_y$ and $u_y = -v_x$ for all $z \in \mathbb{C} \setminus \{1\}$. Hence, f is analytic on $\mathbb{C} \setminus \{1\}$.

(ii) Writing $z = x + iy$, we have

$$f(z) \;=\; \frac{\bar{z}-1}{(z-1)(\bar{z}-1)} \;=\; \frac{1}{z-1}.$$

We see that $f \equiv g(z,\bar{z})$ is free of the second complex variable $\bar{z}$, so $f_{\bar{z}} = 0$. Hence, f is analytic on $\mathbb{C} \setminus \{1\}$.

Problem 2 Compute the contour integral

$$\oint_C \frac{z \exp z}{z^2+1}\,\mathrm{d}z,$$

where C is a circle of radius 2, centered at 0, and oriented counterclockwise.

Solution We carry out the partial fraction decomposition

$$\frac{z}{z^2+1} \;=\; \frac{1}{2}\,\frac{(z+i)+(z-i)}{(z+i)(z-i)} \;=\; \frac{1}{2}\left[\frac{1}{z-i} \;+\; \frac{1}{z+i}\right].$$

Thus, we split our integral

$$I \;=\; \oint_C \frac{z \exp z}{z^2+1}\,\mathrm{d}z \;=\; \frac{1}{2}\oint_C \frac{\exp z}{z-i}\,\mathrm{d}z \;+\; \frac{1}{2}\oint_C \frac{\exp z}{z+i}\,\mathrm{d}z \;=\; \frac{1}{2}I_1 \;+\; \frac{1}{2}I_2.$$

Note that $\exp(z)$ is analytic, and C contains both i and $-i$. Thus, Cauchy's Integral Formula yields

$$I_1 \;=\; 2\pi i \exp(i), \qquad I_2 \;=\; 2\pi i \exp(-i).$$

Using $e^{i\varphi}+e^{-i\varphi} = 2\cos\varphi$ yields

$$I \;=\; 2\pi i \cos 1.$$