MA 1202 : Mathematical Methods I

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Problem 1 Determine the first three terms of each fundamental solution of the ODE

$$y'' - xy = 0,$$

by assuming a power series expansion around the point $x_0 = 2$.

Solution Note that $x_0 = 2$ is a regular point.
We propose a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-2)^n.$$

Hence,

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2}.$$

Plugging this into the ODE,

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} - x \sum_{n=0}^{\infty} a_n (x-2)^n = 0.$$

After some manipulation,

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} - (x-2+2) \sum_{n=0}^{\infty} a_n (x-2)^n = 0,$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} - \sum_{n=0}^{\infty} a_n (x-2)^{n+1} - 2\sum_{n=0}^{\infty} a_n (x-2)^n = 0.$$

Shifting indices,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-2)^n - \sum_{n=1}^{\infty} a_{n-1} (x-2)^n - 2\sum_{n=0}^{\infty} a_n (x-2)^n = 0.$$

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1} - 2a_n\right] (x-2)^n = 0.$$

Setting coefficients to zero, we have

$$\begin{aligned} 2a_2 - 2a_0 &= 0 && n = 0 \\ (n+2)(n+1)a_{n+2} - a_{n-1} - 2a_n &= 0 && n = 1, 2, 3, \dots \end{aligned}$$

Thus,

$$\begin{aligned} a_0 &= a_2 && n = 0 \\ 6a_3 &= a_0 + 2a_1 && n = 1 \\ 12a_4 &= a_1 + 2a_2 && n = 2 \end{aligned}$$

Thus,

$$\begin{aligned} y(x) &= a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + a_4(x-2)^4 + \dots \\ &= a_0 + a_1(x-2) + a_0(x-2)^2 + \left(\frac{a_0}{6} + \frac{a_1}{3}\right)(x-2)^3 + \left(\frac{a_1}{12} + \frac{a_0}{6}\right)(x-2)^4 + \dots \\ &= a_0\left[1 + (x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{6}(x-2)^4 + \dots\right] + a_1\left[(x-2) + \frac{1}{3}(x-2)^3 + \frac{1}{12}(x-2)^4 + \dots\right]. \end{aligned}$$

Hence, our fundamental solutions are

$$y_1(x) \;=\; 1 \;+\; (x-2)^2 \;+\; \frac{1}{6}(x-2)^3 \;+\; \ldots$$
$$y_2(x) \;=\; (x-2) \;+\; \frac{1}{3}(x-2)^3 \;+\; \frac{1}{12}(x-2)^4 \;+\; \ldots$$

Problem 2 Solve the indicial equation, assuming a power series expansion near the regular singulat point $x_0 = 0$, obtained from the ODE

$$x^2 y'' \;+\; \left(x^2 + \frac{5}{36}\right) y \;=\; 0.$$

Find the first three terms for the power series expansion of the fundamental solution corresponding to the largest root of the indicial equation.

Solution We note that the limit

$$\lim_{x\to 0} x^2 \cdot \frac{x^2 + \frac{5}{36}}{x^2} \;=\; \frac{5}{36}$$

is finite.
We propose a solution of the form

$$y(x) \;=\; x^r \sum_{n=0}^{\infty} a_n x^n.$$

Hence,

$$y'(x) \;=\; \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \qquad y''(x) \;=\; \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Plugging this into our equation, we have

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} \;+\; \sum_{n=0}^{\infty} a_n x^{n+r+2} \;+\; \frac{5}{36} \sum_{n=0}^{\infty} a_n x^{n+r} \;=\; 0.$$

Shifting indices and rearranging,

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} \;+\; \sum_{n=2}^{\infty} a_{n-2} x^{n+r} \;+\; \frac{5}{36} \sum_{n=0}^{\infty} a_n x^{n+r} \;=\; 0.$$

$$\left[r(r-1)a_0 + (r+1)r a_1 x + \frac{5}{36}a_0 + \frac{5}{36}a_1 x\right] x^r + \sum_{n=2}^{\infty} \left[(n+r)(n+r-1)a_n + a_{n-2} + \frac{5}{36}a_n\right] x^{n+r} \;=\; 0.$$

Our indicial equation thus is

$$r(r-1) \;+\; \frac{5}{36} \;=\; 0.$$

This has roots

$$r_+ \;=\; \frac{5}{6} \qquad r_- \;=\; \frac{1}{6}.$$

We choose $r = r_+ = 5/6$. Setting coefficients to zero, we have the relations

$$55a_1 \;+\; 5a_1 \;=\; 0,$$
$$(6n+5)(6n-1)a_n \;+\; 36a_{n-2} \;+\; 5a_n \;=\; 0, \qquad n \geq 2.$$

Hence,

$$55a_1 + 5a_1 \;=\; 0 \implies a_1 \;=\; 0$$
$$187a_2 + 36a_0 + 5a_2 \;=\; 0 \implies a_2 \;=\; \frac{3}{16}a_0$$
$$391a_3 + 36a_1 + 5a_3 \;=\; 0 \implies a_3 \;=\; 0$$
$$667a_4 + 36a_2 + 5a_4 \;=\; 0 \implies a_4 \;=\; \frac{3}{56}a_2 \;=\; \frac{9}{896}a_0$$

More generally, $a_n = 3a_{n-2}/(3n^2 + 2n)$ for all $n \geq 2$, so all odd terms vanish. Hence, the corresponding solution is

$$\begin{aligned} y(x) &= x^{r_+}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^2 + \dots) \\ &= a_0 x^{5/6}\left(1 - \frac{3}{16}x^2 + \frac{9}{896}x^4 + \dots\right). \end{aligned}$$

The fundamental solution is thus

$$y_+(x) = x^{5/6}\left(1 - \frac{3}{16}x^2 + \frac{9}{896}x^4 + \dots\right).$$