MA 1202 : Mathematical Methods I

Satvik Saha, 19MS154 April 21, 2020

Problem 1A. Solve the following second order differential equation by the method of undetermined coefficients.

$$y''(x) - 7y'(x) + 12y = 8\sin x + \exp 3x.$$

Solution 1A. We first solve the homogenous differential equation for the complementary solution $y_C(x)$.

$$y''(x) - 7y'(x) + 12y = 0.$$

This is a second order ODE with constant coefficients. Its characteristic polynomial is

$$f(t) = t^2 - 7t + 12 = (t-3)(t-4).$$

The roots of f are clearly 3 and 4. We thus set

$$y_C(x) = A\exp 3x + B\exp 4x.$$

We verify that $\{e^{3x}, e^{4x}\}$ indeed comprise a fundamental set of (linearly independent) solutions of the homogenous ODE by calculating the Wronskian

$$W\left(e^{3x}, e^{4x}\right)(x) = \begin{vmatrix} e^{3x} & e^{4x} \\ 3e^{3x} & 4e^{4x} \end{vmatrix} = e^{7x},$$

which is clearly non-zero over the reals.
To solve for the particular solution $y_P(x)$, using the method of undetermined coefficients, we set

$$y_P(x) = C\sin x + D\cos x + Ex\exp 3x.$$

We choose a guess of xe^{3x} to account for the fact that e^{3x} is part of the complementary solution. Otherwise, the e^{3x} term would vanish on the LHS, but not on the RHS.

Plugging this into the original ODE, we obtain

$$(-C + 7D + 12C)\sin x + (-D - 7C + 12D)\cos x + E(9x + 6 - 21x - 7 + 12x)\exp 3x$$

$$= (11C + 7D)\sin x + (-7C + 11D)\cos x - E\exp 3x$$

Comparing coefficients,

$$\begin{aligned} 11C + 7D &= 8 \\ -7C + 11D &= 0 \\ E &= -1, \end{aligned}$$

which yields $C = 44/85$ and $D = 28/85$.
Hence, the solution to the ODE is given by $y(x) = y_C(x) + y_P(x)$, i.e.

$$y(x) = Ae^{3x} + Be^{4x} + \frac{44}{85}\sin x + \frac{28}{85}\cos x - xe^{3x}. \qquad (\star)$$

Problem 1B. Solve the same differential equation by the method of variation of parameters.
Solution 1B. We have already solved the homogeneous ODE for the complementary solution $y_C(x)$. We set $y_P(x) = u(x)e^{3x} + v(x)e^{4x} = uy_1 + vy_2$. We also stipulate that $u'y_1 + v'y_2 = 0$, i.e. $u' = -v'e^x$. Hence, $y_P'(x) = uy_1' + vy_2'$, and $y_P''(x) = u'y_1' + uy_1'' + v'y_2' + vy_2''$.
Plugging this into the original differential equation, and acknowledging that y_1, y_2 are solutions of the homogenous ODE gives

$$u'y_1' + v'y_2' \;=\; 8\sin x + e^{3x} \;=\; g(x).$$

Together with $u'y_1 + v'y_2 = 0$, we obtain

$$u'(x) \;=\; -\frac{y_2 g(x)}{y_1 y_2' - y_1' y_2} \;=\; -\frac{y_2 g(x)}{W(y_1, y_2)}.$$

Solving this for u yields

$$u(x) \;=\; -\int e^{4x}(8\sin x + e^{3x})e^{-7x}\,dx \;=\; -\int 8\sin x\, e^{-3x} + 1\,dx$$

To calculate the trigonometric integral, we set

$$\begin{aligned}
I \;&=\; \int \cos bx\, e^{ax}\,dx \;+\; i\int \sin bx\, e^{ax}\,dx \\
&=\; \int e^{ibx}e^{ax}\,dx \\
&=\; \frac{1}{a+ib}e^{ax}e^{ibx} \\
&=\; \frac{a-ib}{a^2+b^2}\,(\cos bx + i\sin bx)e^{ax} \\
&=\; \frac{1}{a^2+b^2}\,(a\cos x + b\sin x)e^{ax} \;+\; \frac{1}{a^2+b^2}\,(-b\cos x + a\sin x)ie^{ax}.
\end{aligned}$$

Equating real and imaginary parts gives us the desired result. *We ignore constants of integration as the resulting terms will be absorbed back into the complementary solution.*
Hence,

$$u(x) \;=\; -x \;+\; 8\cdot\frac{1}{10}\cos x\, e^{-3x} \;+\; 8\cdot\frac{3}{10}\sin x\, e^{-3x}$$

Similarly,

$$v'(x) \;=\; -u'(x)e^{-x} \;=\; e^{4x}(8\sin x + e^{3x})e^{-7x}e^{-x}.$$

$$\begin{aligned}
v(x) \;&=\; \int 8\sin x\, e^{-4x} + e^{-x}\,dx \\
&=\; -e^{-x} - 8\cdot\frac{1}{17}\cos x\, e^{-4x} - 8\cdot\frac{4}{17}\sin x\, e^{-4x}.
\end{aligned}$$

Hence,

$$\begin{aligned}
y_P(x) \;&=\; -xe^{3x} + 8\cdot\frac{1}{10}\cos x + 8\cdot\frac{3}{10}\sin x - e^{3x} - 8\cdot\frac{1}{17}\cos x - 8\cdot\frac{4}{17}\sin x \\
&=\; -e^{3x} - xe^{3x} + \frac{28}{85}\cos x + \frac{44}{85}\sin x.
\end{aligned}$$

Putting this together with the complementary solution, and absorbing the extra e^{3x} term, we obtain the same general solution as before,

$$y(x) \;=\; A'e^{3x} \;+\; Be^{4x} \;+\; \frac{44}{85}\sin x \;+\; \frac{28}{85}\cos x \;-\; xe^{3x}. \tag{$\star$}$$

Problem 2. Find the general solution of the equation of motion of a forced oscillator with damping, described by the second order differential equation

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}e^{i\omega t}.$$

Show that the maximum amplitude of the steady state vibration is given by

$$x_{max}\Big|_{\omega=\omega_0} = \frac{F_0}{2m\gamma\omega_0}.$$

Solution 2. We first solve the homogeneous ODE for the complementary solution $x_C(t)$,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 = 0.$$

The characteristic polynomial is given by

$$f(s) = s^2 + 2\gamma s + \omega_0^2.$$

This has roots

$$s_{+/-} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}.$$

Hence, for unequal roots, i.e. $\gamma^2 - \omega_0^2 \neq 0$, we have

$$x_C(t) = \left(Ae^{\sqrt{\gamma^2-\omega_0^2}t} + Be^{-\sqrt{\gamma^2-\omega_0^2}t}\right)e^{-\gamma t}.$$

For equal roots, i.e. $\gamma^2 - \omega_0^2 = 0$, we have

$$x_C(t) = (A + Bt)\,e^{-\gamma t}.$$

We consider 2 cases. *We assume real coefficients, positive γ.*

Case I. $\gamma^2 - \omega_0^2 \neq 0$.
We use the method of undetermined coefficients to construct a particular solution $y_P(t)$.

$$x_P(t) = Ce^{i\omega t}.$$

Plugging this into the original ODE yields

$$(-\omega^2 + 2i\gamma\omega + \omega_0^2)Ce^{i\omega t} = \frac{F_0}{m}e^{i\omega t}.$$

$$C = \frac{F_0}{m((\omega_0^2 - \omega^2) + 2i\gamma\omega)}.$$

Hence,

$$x(t) = \left(Ae^{\sqrt{\gamma^2-\omega_0^2}t} + Be^{-\sqrt{\gamma^2-\omega_0^2}t}\right)e^{-\gamma t} + \frac{F_0}{m((\omega_0^2 - \omega^2) + 2i\gamma\omega)}e^{i\omega t}.$$

Case II. $\gamma^2 - \omega_0^2 = 0$.
The particular solution remains the same. Hence,

$$x(t) = (A + Bt)e^{-\gamma t} + \frac{F_0}{m((\omega_0^2 - \omega^2) + 2i\gamma\omega)}e^{i\omega t}.$$

Note that for real valued coefficients, as $t \to \infty$, $x_C(t) \to 0$. Hence, the steady state response is governed by the particluar solution. Now,

$$\text{Re } x_P(t) = \frac{F_0}{m((\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2)}\left((\omega_0^2 - \omega^2)\cos\omega t + 2\gamma\omega\sin\omega t\right).$$

Setting

$$\phi = \arctan\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right),$$

we obtain

$$\text{Re } x_P(t \to \infty) \;\approx\; \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \cos(\omega t - \phi).$$

Clearly, the amplitude of steady state oscillation is maximized when the denominator is minimized. Setting $\frac{\partial}{\partial \omega}((\omega_0^2 - \omega)^2 + 4\gamma^2\omega^2) = 0$, we have

$$-4(\omega_0^2 - \omega^2)\omega + 8\gamma^2\omega \;=\; 0,$$

$$\omega \;=\; \omega_R \;=\; \sqrt{\omega_0^2 - 2\gamma^2}.$$

This is the resonant angular frequency of the system. Also, assuming real, positive ω_R, i.e. a sufficiently underdamped system,

$$\frac{\partial^2}{\partial \omega^2}((\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2) \;=\; -4\omega_0^2 + 12\omega^2 + 8\gamma^2,$$

$$\frac{\partial^2}{\partial \omega^2}((\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2)\Big|_{\omega=\omega_R} \;=\; 8\omega_0^2 - 16\gamma^2 \;=\; 8\omega_R^2 > 0.$$

Hence, the maximum amplitude at steady state is

$$x_{\text{max}} \;=\; \frac{F_0}{2m\gamma\sqrt{\omega_0^2 - \gamma^2}}.$$

For very weak damping, $\omega_R \to \omega_0$, and thus

$$x_{\text{max}} \;\approx\; \frac{F_0}{2m\gamma\omega_0}.$$