MA 1201 : Mathematics II

Satvik Saha, 19MS154 June 19, 2020

Section 2.5 (Distance Preserving Maps)

Problem 1. Construct a rotation $D_{x,\phi}$ which maps $(1,2)$ and $(4,6)$ respectively onto $(5,2)$ and $(8,-2)$ respectively.

Solution 1. Let the points $P = (1,2)$ and $Q = (4,6)$ be mapped to $P' = (5,2)$ and $Q' = (8,-2)$ respectively. Consider the vector $u = PQ = (3,4)$ in $\mathbb{R}^2$. Under the isometry $D_{x,\phi}$, this gets transformed into the vector $v = P'Q' = (3,-4)$. The angle by which u rotates into v must be precisely the angle ϕ. Thus, $\phi = -2\arccos(3/5)$.

Let $x = (x_1, x_2)$ be the center of rotation. Thus, we must have equal distances $Px = P'x$ and $Qx = Q'x$. The first forces $x_1 = 3$. Thus, from the second condition, we must have $(4-3)^2 + (6-x_2)^2 = (8-3)^2 + (-2-x_2)^2$, which rearranges to $(6-x_2)^2 - (-2-x_2)^2 = 24$. Using the difference of squares, $(4-2x_2)(8) = 24$, thus $x_2 = \frac{1}{2}$. Hence, $x = (3, \frac{1}{2})$.

Thus, the required isometry is $D_{(3,1/2),-2\arccos(3/5)}$.

Problem 2.

(i) Give the coordinate representation of $D_{(1,6),\pi/6}$.

(ii) Give the coordinate representation of the reflection in the line $L_{1,2,-1}$.

(iii) Find an x so that $D_{(3,2),\theta} = D_\theta \circ T_x$, where θ is such that $\cos\theta = 3/5$ and $\sin\theta = 4/5$.

Solution 2.

(i) Note that $D_{x,\phi} = T_x \circ D_\phi \circ T_{-x}$. Setting $x = (1,6)$ and $\phi = \pi/6$, we thus construct the following maps. *Note that* $\cos\phi = \sqrt{3}/2$ *and* $\sin\phi = 1/2$.

$$\begin{aligned}
T_{-x} \colon \mathbb{R}^2 \to \mathbb{R}^2, &\quad (\xi_1, \xi_2) \mapsto (\xi_1 - 1,\, \xi_2 - 6)\,, \\
D_\phi \circ T_{-x} \colon \mathbb{R}^2 \to \mathbb{R}^2, &\quad (\xi_1, \xi_2) \mapsto \left(\frac{\sqrt{3}}{2}(\xi_1 - 1) - \frac{1}{2}(\xi_2 - 6)\,,\, \frac{1}{2}(\xi_1 - 1) + \frac{\sqrt{3}}{2}(\xi_2 - 6)\right), \\
T_x \circ D_\phi \circ T_{-x} \colon \mathbb{R}^2 \to \mathbb{R}^2, &\quad (\xi_1, \xi_2) \mapsto \left(\frac{\sqrt{3}}{2}(\xi_1 - 1) - \frac{1}{2}(\xi_2 - 6) + 1,\, \frac{1}{2}(\xi_1 - 1) + \frac{\sqrt{3}}{2}(\xi_2 - 6) + 6\right).
\end{aligned}$$

Thus, the desired isometry is

$$D_{x,\phi} \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (\xi_1, \xi_2) \mapsto \left(\frac{\sqrt{3}}{2}\xi_1 - \frac{1}{2}\xi_2 + \frac{1}{2}(8 - \sqrt{3}),\, \frac{1}{2}\xi_1 + \frac{\sqrt{3}}{2}\xi_2 + \frac{1}{2}(11 - 6\sqrt{3})\right).$$

(ii) The given line $L = L_{1,2,-1}$ is described by

$$\xi_1 + 2\xi_2 - 1 \;=\; 0.$$

Its perpendicular distance from the origin is simply $d = 1/\sqrt{1^2 + 2^2} = 1/\sqrt{5}$, along the vector $\hat{n} = (1,2)/\sqrt{1^2 + 2^2} = (1/\sqrt{5}, 2/\sqrt{5})$. Thus, the reflection of the origin lies at $2d\hat{n} = (2/5, 4/5)$.

Now, note that the desired reflection is an isometry, and hence is a mapping of the form

$$R_L\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (\xi_1, \xi_2) \mapsto (a_1\xi_1 + b_1\xi_2 + c_1,\, a_2\xi_1 + b_2\xi_2 + c_2).$$

We have already established that $R_L(0,0) = (2/5, 4/5)$, hence $c_1 = 2/5$ and $c_2 = 4/5$. Now, we simply choose two other points on the line L, say $(0, 1/2)$ and $(1, 0)$, which must be fixed points under the reflection. Thus, $b_1/2 + 2/5 = 0$, $b_2/2 + 4/5 = 1/2$, $a_1 + 2/5 = 1$ and $a_2 + 4/5 = 0$. Hence, we obtain

$$R_L\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (\xi_1, \xi_2) \mapsto \left(\frac{3}{5}\xi_1 - \frac{4}{5}\xi_2 + \frac{2}{5},\, -\frac{4}{5}\xi_1 - \frac{3}{5}\xi_2 + \frac{4}{5}\right).$$

(iii) The given isometry is $D_\theta \circ T_x$. Since this is to be equivalent to $D_{(3,2),\theta}$, this isometry must have the fixed point $(3, 2)$, the center of rotation. Thus, $(D_\theta \circ T_x)(3, 2) = (3, 2)$. Applying $D_{-\theta}$ on both sides and using $D_{-\theta} \circ D_\theta = \mathrm{Id}$, we have

$$T_x(3,2) = D_{-\theta}(3,2) = \left(\frac{3}{5}\cdot 3 + \frac{4}{5}\cdot 2, -\frac{4}{5}\cdot 3 + \frac{3}{5}\cdot 2\right) = \left(\frac{17}{5}, -\frac{6}{5}\right) = \left(3 + \frac{2}{5}, 2 - \frac{16}{5}\right).$$

By comparison with $T_x(3, 2) = (3 + x_1, 2 + x_2)$, we must have $x = (2/5, -16/5)$.

Note that we have used $D_{-\theta}(\xi_1, \xi_2) = (\xi_1 \cos\theta + \xi_2 \sin\theta, -\xi_1 \sin\theta + \xi_2 \cos\theta)$.

Problem 3. Determine the geometric forms of the mappings

(i) $(\xi_1, \xi_2) \mapsto (\frac{8}{17}\xi_1 + \frac{15}{17}\xi_2 - 1, \frac{15}{17}\xi_1 - \frac{8}{17}\xi_2 + 3)$.

(ii) $(\xi_1, \xi_2) \mapsto (\frac{3}{5}\xi_1 + \frac{4}{5}\xi_2 - 10, -\frac{4}{5}\xi_1 + \frac{3}{5}\xi_2 - 1)$.

Solution 3.

(i) The transformation matrix of the given mapping is $\begin{bmatrix} 8/17 & 15/17 \\ 15/17 & -8/17 \end{bmatrix}$, which clearly represents a reflection. Thus, the given mapping is a glide reflection. Note that the determinant of the matrix is -1.

(ii) The transformation matrix of the given mapping is $\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$, which clearly represents a rotation by the angle $\theta = -\arccos\frac{3}{5}$. The point about which the rotation takes place is the fixed point of the isometry, i.e. we solve

$$\begin{aligned} \frac{3}{5}\xi_1 + \frac{4}{5}\xi_2 - 10 &= \xi_1 \\ -\frac{4}{5}\xi_1 + \frac{3}{5}\xi_2 - 1 &= \xi_2 \end{aligned}$$

This yields $x_0 = (-6, 19/2)$. Thus, the given mapping is the (clockwise) rotation $D_{x_0,\theta}$. Note that the determinant of the matrix is $+1$.

Problem 4. Show that if ABC and PQR are triangles in $\mathbb{R}^2$ such that $|AB| = |PQ|$, $|BC| = |QR|$ and $|CA| = |RP|$, then there is an isometry f on the plane which maps A, B, C onto P, Q, R respectively. When is such an f unique?

Solution 4. We show the existence of f by construction. Let $v = AP$ be the vector stretching from A to P. Thus, applying the isometry T_v maps the points (A, B, C) to (P, B_1, C_1). Now, the isometry preserves distances, so $|PB_1| = |AB| = |PQ|$. This means that B_1 and Q lie on the same circle centred at P, with radius $|AB|$. Hence, there exists an angle θ between PB_1 and PQ such that the rotation $D_{P,\theta}$ maps the points (P, B_1, C_1) to (P, Q, C_2). Again, note that $|PC_2| = |PC_1| = |AC| = |PR|$, and $|QC_2| = |B_1C1| = |BC| = |QR|$. Hence, C_2 lies on the intersection of the circles centred at P and Q with radii $|PR|$ and $|QR|$ respectively. These circles must intersect, since we know that the point R exists. If these circles intersect once, this forces $C_2 = R$ and we are done. Otherwise, the circles intersect twice.

Either $C_2 = R$, or C_2 and R are reflections of each other in the line L containing the segment PQ. Hence, the application of a final reflection R_L maps (P, Q, C_2) to (P, Q, R). Since the composition of isometries is also an isometry, we have $f = R_L \circ D_{P,\theta} \circ T_v$ (or $f = D_{P,\theta} \circ T_v$ if $C_2 = R$) which is the isometry we seek.

Note that if A, B, and C are collinear, then so are P, Q and R. In this case, the isometry $f' = R_L \circ f$ is a different isometry which also has the desired properties, since P, Q, R all lie on the line L and hence are fixed points of R_L.
Otherwise, let f_1 and f_2 be two isometries which map (A, B, C, X) to (P, Q, R, X_1) and (P, Q, R, X_2) respectively, where X is an arbitrary point in $\mathbb{R}^2$. Clearly, if X is one of A, B or C, we must have $X_1 = X_2$. If not, note that we must have $|AX| = |PX_1| = |PX_2|$, $|BX| = |QX_1| = |QX_2|$ and $|CX| = |RX_1| = |RX_2|$, so P, Q and R are all equidistant from X_1 and X_2. If $X_1 \neq X_2$, this forces P, Q, R to lie on the same line (the locus of points equidistant from two points is a line), which is a contradiction. Hence, $X_1 = X_2$ for all $X \in \mathbb{R}^2$. This means that we must have $f_1 = f_2$.

Thus, the isometry between (A, B, C) and (P, Q, R) is unique iff A, B and C are noncollinear.

(Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be two different points in $\mathbb{R}^2$. If $X = (x_1, x_2)$ is to be equidistant from P and Q, then $(p_1 - x_1)^2 + (p_2 - x_2)^2 = (q_1 - x_1)^2 + (q_2 - x_2^2)$. Rearranging, $x_1^2 + x_2^2 - 2p_1x_1 - 2p_2x_2 + p_1^2 + p_2^2 = x_1^2 + x_2^2 - 2q_1x_1 - 2q_2x_2 + q_1^2 + q_2^2$, i.e. $2(q_1 - p_1)x_1 + 2(q_2 - p_2)x_2 = q_1^2 - p_1^2 + q_2^2 - p_2^2$. Since $p_1 \neq q_1$ or $q_2 \neq q_1$, this is the equation of a line.)

Section 2.6 (Conic Sections)

Problem 1. Describe the geometric form of the following curves.

(i) $\xi_1^2 + 6\xi_1\xi_2 + 9\xi_2^2 + 5\xi_1 + 2\xi_2 + 11 = 0$.

(ii) $4\xi_1^2 + 4\xi_1\xi_2 - 10\xi_1 + 8\xi_2 + 15 = 0$.

(iii) $\xi_1^2 + \xi_1\xi_2 + \xi_2^2 = 3$.

(iv) $5\xi_1^2 + 6\xi_1\xi_2 + 5\xi_2^2 - 256 = 0$.

(v) $\xi_1^2 - 2\xi_1\xi_2 + \xi_2^2 = 9$.

Solution 1.

(i) We have

$$Q\colon (\xi_1, \xi_2) \mapsto \xi_1^2 + 6\xi_1\xi_2 + 9\xi_2^2 + 5\xi_1 + 2\xi_2 + 11 \;=\; (f(x)\,|\,x) + 2(b\,|\,x) + 11.$$

The matrix of the quadratic part is $A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$, whose eigenvalues satisfy $(\lambda - 1)(\lambda - 9) = 9$. Thus, the only non-zero eigenvalue is $\lambda = 10$, whose corresponding eigenvector $x_1 = (x_{11}, x_{12})$ satisfies $(x_{11} + 3x_{12}, 3x_{11} + 9x_{12}) = (10x_{11}, 10x_{12})$. We choose $x_1 = (1, 3)/\sqrt{1^2 + 3^2} = (1/\sqrt{10}, 3/\sqrt{10})$. The vector orthogonal to x_1 is given by $x_2 = D_{\pi/2}(x_1) = (-3/\sqrt{10}, 1/\sqrt{10})$. Thus, changing basis to x_1, x_2, we have

$$\begin{aligned} Q(\eta_1 x_1 + \eta_2 x_2) &= \lambda\eta_1^2 + \frac{1}{\sqrt{10}}(5\cdot 1 - 3\cdot 2)\eta_1 + \frac{1}{\sqrt{10}}(-5\cdot 3 + 2\cdot 1)\eta_2 + 11 \\ &= 10\eta_1^2 + \frac{11}{\sqrt{10}}\eta_1 - \frac{13}{\sqrt{10}}\eta_2 + 11 \\ &= 10\left(\eta_1 + \frac{11}{20\sqrt{10}}\right)^2 - \frac{13}{\sqrt{10}}\left(\eta_2 - \frac{4279\sqrt{10}}{5200}\right) \end{aligned}$$

Thus, the given curve is the parabola

$$10\zeta_1^2 - \frac{13}{\sqrt{10}}\zeta_2 \;=\; 0.$$

This is also apparent upon noting that $\det(A) = 0$, which indicates one zero eigenvalue.

(ii) We have

$$Q\colon (\xi_1, \xi_2) \mapsto 4\xi_1^2 + 4\xi_1\xi_2 - 10\xi_1 + 8\xi_2 + 15 = 0 \;=\; (f(x)\,|\,x) + 2(b\,|\,x) + 11.$$

The matrix of the quadratic part is $A = \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}$, whose eigenvalues satisfy $(\lambda - 4)\lambda = 4$. Thus, we have the eigenvalues $\lambda_{1,2} = 2 \pm 2\sqrt{2}$. The first eigenvector thus satisfies $2x_{11} = (2 + 2\sqrt{2})x_{12}$, so we choose $x_1 = (1 + \sqrt{2}, 1)/\sqrt{(1 + \sqrt{2})^2 + 1^2} = (1 + \sqrt{2}, 1)/\sqrt{4 + 2\sqrt{2}}$ and $x_2 = D_{\pi/2}(x_1) = (-1, 1 + \sqrt{2})/\sqrt{4 + 2\sqrt{2}}$. Thus, changing basis to x_1, x_2, we have

$$\begin{aligned}
Q(\eta_1 x_1 + \eta_2 x_2) \;&=\; \lambda_1\eta_1^2 + \lambda_2\eta_2^2 + \frac{1}{\sqrt{4 + 2\sqrt{2}}}(-10 \cdot (1 + \sqrt{2}) + 8)\eta_1 + \frac{1}{\sqrt{4 + 2\sqrt{2}}}(10 \cdot 1 + 8(1 + \sqrt{2}))\eta_2 + 15 \\
&=\; (2 + 2\sqrt{2})\eta_1^2 + (2 - 2\sqrt{2})\eta_2^2 + \frac{(-2 - 10\sqrt{2})\eta_1 + (18 + 8\sqrt{2})\eta_2}{\sqrt{4 + 2\sqrt{2}}} + 15 \\
&=\; (2\sqrt{2} + 2)\left(\eta_1 - \frac{1 + 5\sqrt{2}}{(2 + \sqrt{2})\sqrt{4 + \sqrt{2}}}\right)^2 - (2\sqrt{2} - 2)\left(\eta_2 + \frac{9 + 4\sqrt{2}}{(2 - \sqrt{2})\sqrt{4 + \sqrt{2}}}\right)^2 \\
&\qquad + 15 - \frac{(1 + 5\sqrt{2})^2}{(2 + 2\sqrt{2})^2(4 + 2\sqrt{2})} - \frac{(9 + 4\sqrt{2})^2}{(2 - 2\sqrt{2})^2(4 + 2\sqrt{2})}
\end{aligned}$$

Thus, the given curve is the hyperbola

$$(2\sqrt{2} + 2)\zeta_1^2 \;-\; (2\sqrt{2} - 2)\zeta_2^2 + k \;=\; 0,$$

for non-zero k. This is also apparent upon noting that $\det(A) < 0$, which indicates eigenvalues of opposing sign.

(iii) We have

$$Q\colon (\xi_1, \xi_2) \mapsto \xi_1^2 + \xi_1\xi_2 + \xi_2^2 - 3 \;=\; (f(x)\,|\,x) - 3.$$

The matrix of the quadratic part is $A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$, whose eigenvalues satisfy $(\lambda - 1)^2 = 1/4$. Thus, we have the eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = 3/2$. The first eigenvector x_1 satisfies $x_{11} + x_{12}/2 = (1/2)x_{11}$, so we choose $x_1 = (1, 1)/\sqrt{2}$ and $x_2 = D_{\pi/2}(x_1) = (-1, 1)/\sqrt{2}$. Thus, changing basis to x_1, x_2, we have

$$\begin{aligned}
Q(\eta_1 x_1 + \eta_2 x_2) \;&=\; \lambda_1\eta_1^2 + \lambda_2\eta_2^2 - 3 \\
&=\; \frac{1}{2}\eta_1^2 \;+\; \frac{3}{2}\eta_2^2 - 3
\end{aligned}$$

Thus, the given curve is the ellipse

$$\frac{1}{2}\zeta_1^2 \;+\; \frac{3}{2}\zeta_2^2 \;-\; 3 \;=\; 0.$$

This is also apparent upon noting that $\det(A) > 0$, which indicates eigenvalues of the same sign.

(iv) We have

$$Q\colon (\xi_1, \xi_2) \mapsto 5\xi_1^2 + 6\xi_1\xi_2 + 5\xi_2^2 - 256 \;=\; (f(x)\,|\,x) - 256.$$

The matrix of the quadratic part is $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$, whose eigenvalues satisfy $(\lambda - 5)^2 = 9$. Thus, we have the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 8$. The first eigenvector x_1 satisfies $5x_{11} + 3x_{12} = 2x_{11}$, so we choose $x_1 = (1, -1)/\sqrt{2}$ and $x_2 = D_{\pi/2}(x_1) = (1, 1)/\sqrt{2}$. Thus, changing basis to x_1, x_2, we have

$$\begin{aligned}
Q(\eta_1 x_1 + \eta_2 x_2) \;&=\; \lambda_1\eta_1^2 + \lambda_2\eta_2^2 - 256 \\
&=\; 2\eta_1^2 \;+\; 8\eta_2^2 - 256
\end{aligned}$$

Thus, the given curve is the ellipse

$$2\zeta_1^2 \;+\; 8\zeta_2^2 \;-\; 256 \;=\; 0.$$

This is also apparent upon noting that $\det(A) > 0$, which indicates eigenvalues of the same sign.

(v) Note that the given curve is of the form

$$(\xi_1 - \xi_2)^2 - 3^2 \;=\; 0.$$

Using the difference of squares and separating factors, we obtain the pair of parallel straight lines

$$\begin{aligned} \xi_1 - \xi_2 + 3 &= 0, \\ \xi_1 - \xi_2 - 3 &= 0. \end{aligned}$$

Note that the transformation matrix of the quadratic has a determinant of zero. Thus, these parallel straight lines may be interpreted as a degenerate parabola.

Problem Set 6.1 (Introduction to Eigenvalues)

Problem 6. Find the eigenvalues of A, B, AB and BA.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

(i) Are the eigenvalues of AB equal to the eigenvalues of A times the eigenvalues of B?

(ii) Are the eigenvalues of AB equal to the eigenvalues of BA?

Solution 6. The eigenvalues of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are simply roots of the characteristic polynomial

$$f(t) = (a - t)(d - t) - bc = t^2 - (a + d)t + (ad - bc).$$

Thus, we calculate

$$\begin{aligned} f_A(t) &= t^2 - 2t + 1 = 0, & \lambda_A &= 1. \\ f_B(t) &= t^2 - 2t + 1 = 0, & \lambda_B &= 1. \\ f_{AB}(t) &= t^2 - 4t + 1 = 0, & \lambda_{AB} &= 2 \pm \sqrt{3}. \\ f_{BA}(t) &= t^2 - 4t + 1 = 0, & \lambda_{BA} &= 2 \pm \sqrt{3}. \end{aligned}$$

(i) Note that the eigenvalues of AB are *not* the product of eigenvalues of A and B.

(ii) The eigenvalues of AB in this particular case are indeed the eigenvalues of BA. However, they do not share the same corresponding eigenvectors (this is obvious when solving $(AB)v = (BA)v = \lambda v$, which forces $v = 0$).

Problem 14. Solve $\det(Q - \lambda I) = 0$ by the quadratic formula to reach $\lambda = \cos\theta \pm i \sin\theta$.

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Note that Q rotates the xy plane by the angle θ, with no real λ's. Find the eigenvectors of Q by solving $(Q - \lambda I)x = 0$.

Solution 14. Using the identity $\sin^2\theta + \cos^2\theta = 1$, we have

$$f(t) \;=\; t^2 - (2\cos\theta)t + (\cos^2\theta + \sin^2\theta) = 0, \qquad \lambda_\pm \;=\; \frac{1}{2}(2\cos\theta \pm \sqrt{4\cos^2\theta - 4}) = \cos\theta \pm i\sin\theta.$$

Clearly, the eigenvalues $\lambda_\pm$ are not real (except when $\theta = n\pi$, which corresponds either to a half turn, or the identity).

To solve for the eigenvectors,

$$Q - \lambda_\pm I \;=\; \begin{bmatrix} \mp i\sin\theta & -\sin\theta \\ \sin\theta & \mp i\sin\theta \end{bmatrix} \;=\; \sin\theta \begin{bmatrix} \mp i & 1 \\ 1 & \mp i \end{bmatrix}$$

Thus, for eigenvalue $\lambda_+ = \cos\theta + i\sin\theta$,

$$(Q - \lambda_+ I)v_+ \;=\; 0, \qquad \sin\theta \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} v_{+1} \\ v_{+2} \end{bmatrix} \;=\; 0, \qquad v_{+1} \;=\; iv_{+2}.$$

For eigenvalue $\lambda_- = \cos\theta - i\sin\theta$,

$$(Q - \lambda_- I)v_- \;=\; 0, \qquad \sin\theta \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \begin{bmatrix} v_{-1} \\ v_{-2} \end{bmatrix} \;=\; 0, \qquad v_{-1} \;=\; -iv_{-2}.$$

Thus, we choose

$$v_+ \;=\; \begin{bmatrix} i \\ 1 \end{bmatrix}, \qquad v_- \;=\; \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Problem 17. The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has } \det(A - \lambda I) = \lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

Using the quadratic formula, find the eigenvalues. Find their sum. If $\lambda_1 = 3$ and $\lambda_2 = 4$, find $\det(A - \lambda I)$.

Solution 17. Using the quadratic formula, we write the roots of the given characteristic polynomial as follows.

$$\lambda_\pm \;=\; \frac{1}{2}\left((a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}\right) \;=\; \frac{1}{2}\left((a+d) \pm \sqrt{(a-d)^2 + 4bc}\right).$$

Their sum $\lambda_+ + \lambda_- = a + d = \text{trace}(A)$.

If $\lambda_1 = 3$ and $\lambda_2 = 4$, then note that these are roots of $\det(A - \lambda I)$. Thus,

$$\det(A - \lambda I) \;=\; (\lambda - 3)(\lambda - 4) \;=\; \lambda^2 - 7\lambda + 12.$$

Problem 25. Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $x_1, \ldots, x_n$. Then, show that $A = B$.

Solution 25. Note that since A and B have n eigenvalues and independent eigenvectors, we must have $\dim(A) = \dim(B) = n$. Also note that since all eigenvectors $v_i \in V$ are independent, they comprise a basis of the n dimensional vector space V. Let $x \in V$ be arbitrary. Thus, x has a unique representation in the basis $\{v_1, \ldots, v_n\}$. For scalars $c_1, \ldots, c_n$,

$$x = c_1 v_1 + \cdots + c_n v_n.$$

Now, we compute the products

$$\begin{aligned} Ax \;&=\; A(c_1v_1 + \cdots + c_nv_n) \;=\; c_1(Av_1) + \cdots + c_n(Av_n) \;=\; c_1\lambda_1 v_1 + \cdots + c_n\lambda_n v_n. \\ Bx \;&=\; B(c_1v_1 + \cdots + c_nv_n) \;=\; c_1(Bv_1) + \cdots + c_n(Bv_n) \;=\; c_1\lambda_1 v_1 + \cdots + c_n\lambda_n v_n. \end{aligned}$$

We use the fact that $Av_i = \lambda_i v_i = Bv_i$. Thus, $Ax = Bx$ for all $x \in V$. Hence, we must have $A = B$.

Specifically, we let x_i be such that its i^{th} coordinate is 1 and all other entries are 0. Then $Ax_i = A_i$ and $Bx_i = B_i$, where A_i and B_i are the i^{th} columns of A and B. Thus, since $Ax_i = Bx_i$ for all $i = 1, \ldots, n$, we see that A and B are equal column by column. Hence, $A = B$.