MA 1201 : Mathematics II

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Excercise 5 (S.K. Mapa)

Problem 1. Prove that the following sets are enumerable.

(i) The set of all integral multiples of 5.

(ii) The set of all integral powers of 2.

(iii) The set of all ordered pairs $\{(m, n) : m \in \mathbb{Z}, n \in \mathbb{Z}\}$.

Solution 1. We first supply the bijection $f : \mathbb{Z} \to \mathbb{N}$, defined as

$$f(k) \;=\; \begin{cases} 2k+1 & k \geq 0 \\ -2k & k < 0 \end{cases}, \qquad \text{for all } k \in \mathbb{Z}.$$

Clearly, f is injective, since for $f(k_1) = f(k_2) = n$, where $k_1, k_2 \in \mathbb{Z}$, k_1 and k_2 must either both be non-negative or both be negative, since n is either odd or even respectively. This directly implies that $k_1 = k_2$. Again, f is surjective, since for arbitrary $n \in \mathbb{N}$, we have $n = f((n-1)/2)$ if n is odd, and $n = f(-n/2)$, if n is even. This proves that f is both injective and surjective, hence, f is bijective, with a well defined inverse $f^{-1} : \mathbb{N} \to \mathbb{Z}$. Hence, $\mathbb{Z}$ is enumerable.

(i) Let $S = \{5k : k \in \mathbb{Z}\}$ be the set of all integral multiples of 5. Clearly, $S \subseteq \mathbb{Z}$, since for any $a = 5k \in S$, where $k \in \mathbb{Z}$, we must have $a \in \mathbb{Z}$. Hence, S is an infinite subset of the enumerable set $\mathbb{Z}$, and is hence enumerable.

(ii) Let $T = \{2^k : k \in \mathbb{Z}\}$ be the set of all integral powers of 2. We supply the bijection $h : \mathbb{Z} \to T$, $h(k) = 2^k$ for all $k \in \mathbb{Z}$. Clearly, for $h(k_1) = h(k_2)$, we must have $2^{k_1} = 2^{k_2} \implies k_1 = k_2$, and for all $s \in T$, there exists $k \in \mathbb{Z}$ such that $s = 2^k$. Thus, we have the bijection $(f \circ h^{-1}) : T \to \mathbb{N}$, thus proving that T and $\mathbb{N}$ are equipotent. Hence, T is enumerable.

(iii) We supply the bijection $g : \mathbb{Z}^2 \to \mathbb{N}^2$,

$$g(k_1, k_2) \;=\; (f(k_1),\, f(k_2)), \qquad \text{for all } (k_1, k_2) \in \mathbb{Z}^2.$$

Clearly, g is injective since for $g(k_1, k_2) = g(k_3, k_4)$, we must have $f(k_1) = f(k_3)$ and $f(k_2) = f(k_4)$. The bijectivity of f guarantees that $(k_1, k_2) = (k_3, k_4)$. Also, g must be bijective since for any $(m, n) \in \mathbb{N}^2$, we set $k_1 = f^{-1}(m)$, $k_2 = f^{-1}(n)$, so that $g(k_1, k_2) = (m, n)$ and $(k_1, k_2) \in \mathbb{Z}^2$.

Now, $\mathbb{N}^2$ is enumerable. To show this, we supply the injection $h : \mathbb{N}^2 \to \mathbb{N}$,

$$h(m, n) \;=\; 2^m 3^n, \qquad \text{for all } (m, n) \in \mathbb{N}^2.$$

Clearly, h is injective since for $h(m_1, n_1) = h(m_2, n_2)$, we have $2^{m_1} 3^{n_1} = 2^{m_2} 3^{n_2} = k$, and thus $(m_1, n_1) = (m_2, n_2)$ by the unique factorisation of k. Thus, setting $h(\mathbb{N}^2) = B$, we have $B \subseteq \mathbb{N}$, and the bijection $h' : \mathbb{N}^2 \to B$. Also, B is infinite. Hence, B is enumerable. The bijections h' and g show that $\mathbb{N}^2$ must be enumerable, hence $\mathbb{Z}^2$ must also be enumerable.

$$\mathbb{Z}^2 \xrightarrow{\quad g \quad} \mathbb{N}^2 \xrightarrow{\quad h' \quad} B \subseteq \mathbb{N}.$$

Problem 2. Prove that the set of all closed and bounded intervals of the form $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ with rational endpoints a, b is enumerable.

Solution 2. Let S be the set of all closed and bounded intervals $[a, b]$, where $a \leq b$, $a, b \in \mathbb{Q}$. We construct $f : S \to \mathbb{Q}^2$,

$$f([a, b]) = (a, b), \qquad \text{for all } a \leq b, \text{ where } a, b \in \mathbb{Q}.$$

Clearly, f is injective, since for $f([a, b]) = f([c, d])$, we must have $(a, b) = (c, d)$, hence $[a, b] = [c, d]$. Thus, $f' : S \to T$ is a bijection, where $f(S) = T \subseteq \mathbb{Q}^2$, so S and T are equipotent. Also, T is clearly infinite, so T is an infinite subset of $\mathbb{Q}^2$.
Now, $\mathbb{Q}$ is enumerable, so there exists a bijection $g : \mathbb{Q} \to \mathbb{N}$. We thus construct the bijection $h : \mathbb{Q}^2 \to \mathbb{N}^2$, $h(a, b) = (g(a), g(b))$, for all $a, b \in \mathbb{Q}$. We have already shown that $\mathbb{N}^2$ is enumerable. Hence, $\mathbb{Q}^2$ is enumerable, so T is enumerable, and therefore S is enumerable.

Problem 3. Prove that the set of all circles in a plane having rational radii and centres with rational coordinates is enumerable.

Solution 3. Let C be the set of all such circles. Clearly, a circle is fully determined by its radius and the x and y coordinates of its centre. Hence, we have the injection $f : C \to \mathbb{Q}^3$, $f(C_{rxy}) = (r, x, y)$, for all circles $C_{rxy} \in C$. Here, C_{rxy} is the circle with radius $r \geq 0$, centred at (x, y). Note that f is an injection since (r, x, y) can describe at most one circle. Thus, $f' : C \to S$ is a bijection, where $S = f(C) \subseteq \mathbb{Q}^3$. Note that S is an infinite set, since C is infinite.
Now, we know that $\mathbb{Q}^2$ and $\mathbb{Q}$ are both enumerable, hence equipotent, so there exists a bijection $g : \mathbb{Q}^2 \to \mathbb{Q}$. We thus construct the bijection $h : \mathbb{Q}^3 \to \mathbb{Q}$, $h(a, b, c) = g(g(a, b), c)$. This means that $\mathbb{Q}^3$ is also enumerable. Hence, S must also be enumerable, so C is also enumerable.

Problem 4. Prove that the Cartesian product of two enumerable sets is enumerable.

Solution 4. Let $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ and $B = \{b_1, b_2, \ldots, b_n, \ldots\}$ be two arbitrary enumerable sets. We let $A_i = \{(a_i, b_1), (a_i, b_2), \ldots\}$ for all $i \in \mathbb{N}$. Clearly, A_i is enumerable, beacuse of the existence of the bijection $f : \mathbb{N} \to A_i$, $f(n) = (a_i, b_n)$. Also, A_i is infinite since B is infinite.
Now, the Cartesian product of A and B can be written as $A \times B = \bigcup_{i \in \mathbb{N}} A_i$. This is the union of the inifinitely many enumerable sets A_i, and is hence enumerable.

Problem 5. Let S be an enumerable set and T be an infinite non-enumerable subset of $\mathbb{R}$. Show that

(i) $S \cup T$ is non-enumerable.

(ii) $S \cap T$ is at most enumerable.

(iii) $S - T$ is at most enumerable.

(iv) $T - S$ is non-enumerable.

Solution 5.

(i) Assume that $S \cup T$ is enumerable. This would imply that $T \subseteq S \cup T$, an infinite subset of the enumerable set $S \cup T$, is enumerable. This is a contradiction, hence, $S \cup T$ is non-enumerable.

(ii) Let $S = \{x_1, x_2, \ldots, x_n, \ldots\}$. We construct the mapping $f : S \cap T \to \mathbb{N}$,

$$f(x) = n, \qquad \text{for all } x = x_n \in S \cap T.$$

This function is well defined, since every element $x \in S \cap T$ must belong to S. Also, if $f(y) = f(z) = n$, then we must have $y = x_n = z$, so f is an injection. Thus, $f' : S \cap T \to V$ is a bijection, where $V = f(S \cap T) \subseteq \mathbb{N}$. V is a subset of the countable subset $\mathbb{N}$, hence is countable. Therefore, $S \cap T$, which is equipotent to V, must also be countable, i.e. at most enumerable.

(iii) Let $S = \{x_1, x_2, \ldots, x_n, \ldots\}$. We again construct the mapping $f : S - T \to \mathbb{N}$,

$$f(x) = n, \qquad \text{for all } x = x_n \in S - T.$$

This function is well defined, since every element $x \in S - T$ must belong to S. Also, if $f(y) = f(z) = n$, then we must have $y = x_n = z$, so f is an injection. Thus, $f' : S - T \to U$ is a bijection, where $U = f(S - T) \subseteq \mathbb{N}$. U is a subset of the countable subset $\mathbb{N}$, hence is countable. Therefore, $S - T$, which is equipotent to U, must also be countable, i.e. at most enumerable.

(iv) Assume that $T - S$ is enumerable. Then, the union of the two enumerable sets $T - S$ and S, which is simply $S \cup T$, must also be enumerable. This is a contradiction. Hence, $T - S$ is non-enumerable.

Problem 6. Prove that the sets A and B are equipotent.

(i) $A = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, $B = \{x \in \mathbb{R} : 0 \leq x < 1\}$.

(ii) $A = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, $B = \{x \in \mathbb{R} : a \leq x \leq b\}$.

(iii) $A = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, $B = \{x \in \mathbb{R} : 0 < x < 1\}$.

(iv) $A = \{x \in \mathbb{R} : x \geq 1\}$, $B = \{x \in \mathbb{R} : x > 1\}$.

Solution 1.

(i) We supply the bijection $f \colon [0,1] \to [0,1)$,

$$f(x) \;=\; \begin{cases} \frac{1}{n+1} & x = \frac{1}{n} \text{ for any } n \in \mathbb{N} \\ x & x \neq \frac{1}{n} \text{ for any } n \in \mathbb{N} \end{cases}, \qquad \text{for all } x \in [0,1].$$

Note that $0 \leq f(x) < 1$, so $f([0,1]) \in [0,1)$. Now, if $f(x) = f(y) = z$, either z is of the form $\frac{1}{n+1}$ for $n \in \mathbb{N}$, in which case $x = y = \frac{1}{n}$, or $x = y = z$ otherwise. This, f is injective. In addition, let $z \in [0,1)$ be arbitrary. If z is of the form $\frac{1}{n+1}$ for some $n \in \mathbb{N}$, we have found $f(\frac{1}{n}) = z$, $\frac{1}{n} \in [0,1]$. Otherwise, $f(z) = z$, where $z \in [0,1]$. Hence, $f \colon A \to B$ is a bijection, proving that A and B are equipotent.

Note that there is no $x \in [0,1]$ such that $f(x) = 1$, since $f(1) = \frac{1}{2}$, and for all $y \in [0,1)$, $f(y)$ is either $\frac{1}{n+1} < 1$, or $f(y) = y < 1$.

(ii) We supply the bijection $f \colon [0,1] \to [a,b]$,

$$f(x) \;=\; a + (b-a)x, \qquad \text{for all } x \in [0,1].$$

Clearly, if $f(x) = f(y)$, then $a + (b-a)x = a + (b-a)y \implies x = y$. Also, for arbitrary $z \in [a,b]$, we find $x = (z-a)/(b-a) \in [0,1]$ such that $f(x) = z$. Note that $a \leq z \leq b \implies 0 \leq z - a \leq b - a$. Hence, $f \colon A \to B$ is a bijection, proving that A and B are equipotent.

(iii) We supply the bijection $f \colon (0,1) \to [0,1]$,

$$h(x) \;=\; \begin{cases} 0 & x = \frac{1}{2} \\ 1 & x = \frac{1}{3} \\ \frac{1}{n-2} & x = \frac{1}{n} \text{ for any } n > 3, n \in \mathbb{N} \\ x & x \neq \frac{1}{n} \text{ for any } n \in \mathbb{N} \end{cases}, \qquad \text{for all } x \in (0,1).$$

Note that if $h(x) = h(y) = z$, either $z = 0 \implies x = y = \frac{1}{2}$, or $z = 1 \implies x = y = \frac{1}{3}$, or $z = \frac{1}{n}$ for $n > 3 \implies x = y = \frac{1}{n-2}$, or none of the above, in which case $x = y = z$ again. Thus, h is injective.

In addition, for arbitrary $z \in \{0,1\}$, we find $h(\frac{1}{2}) = 0$, $h(\frac{1}{3}) = 1$. For $z \in (0,1)$, where z is of the form $\frac{1}{n}$ for $n \in \mathbb{N}$, we have $h(\frac{1}{n+2}) = z$. For $z \in (0,1)$, $z \neq \frac{1}{n}$, we have $h(z) = z$. This, h is surjective. Hence, $f \colon B \to A$ is a bijection, proving that A and B are equipotent.

(iv) We supply the bijection $f \colon [1,\infty) \to (1,\infty)$,

$$f(x) \;=\; \begin{cases} x+1 & x \in \mathbb{N} \\ x & x \notin \mathbb{N} \end{cases}, \qquad \text{for all } x \in [1,\infty).$$

Clearly, if $f(x) = f(y) = z$, either z is an integer or not, and in either case, $x = y$, so f is injective. Also, for arbitrary $z \in (1,\infty)$, if z is an integer, we have $f(z-1) = z$, where $z - 1 \geq 1$. Otherwise, $f(z) = z$, where $z > 1$. Thus, f is surjective. Hence, $f \colon A \to B$ is a bijection, proving that A and B are equipotent.

Exercises (Bartle and Sherbert)

Problem 1. Prove that a non-empty set T_1 is finite if and only if there is a bijection from T_1 onto a finite set T_2.

Solution 1. First, let $f\colon T_1 \to T_2$ be a bijection, where T_2 is finite. We claim that T_1 is finite. Since T_1 is non-empty, T_2 cannot be empty. Let T_2 have n elements. The finiteness of T_2 implies the existence of a bijection $g\colon \mathbb{N}_n \to T_2$, where $\mathbb{N}_n = \{1, 2, \dots, n\}$. Hence, we construct the bijection $h\colon \mathbb{N}_n \to T_1$, defined by $h = f^{-1} \circ g$. This proves that T_2 has n elements, and hence is finite.

Now let T_1 be non-empty and finite, with $n \in \mathbb{N}$ elements. Then, we have the bijection $f\colon \mathbb{N}_n \to T_1$, whose inverse $f^{-1}\colon T_1 \to \mathbb{N}_n$ is also a bijection. Setting $T_2 = \mathbb{N}_n$, which is a finite set, we are done.

Problem 2. Prove the following.

(i) If A is a set with $m \in \mathbb{N}$ elements and $C \subseteq A$ is a set with 1 element, then $A \setminus C$ is a set with $m - 1$ elements.

(ii) If C is an infinite set and B is a finite set, then $C \setminus B$ is an infinite set.

Solution 2.

(i) Let $f\colon \mathbb{N}_m \to A$ and $g\colon \{1\} \to C$ be bijections. Let $c \in C$ be the element $c = g(1)$. Let $k = f^{-1}(c)$. Then, we construct the bijection $h\colon \mathbb{N}_{m-1} \to A \setminus C$,

$$h(n) = \begin{cases} f(n) & n = 1, 2, \dots, k-1 \\ f(n+1) & n = k, k+1, \dots, m-1 \end{cases}.$$

Note that $f(n) \in A$ for all $n \in \mathbb{N}_m$, and if $n \neq k$, then $f(n) \neq f(k)$ from the injectivity of f, so $f(n) \notin C$. Hence, the function is well defined.

We now show that this is indeed a bijection. Let $a \neq b$, where $a, b \in \mathbb{N}_{m-1}$. If $a < k$ and $b < k$, then $h(a) = f(a)$, $h(b) = f(b)$, so $h(a) \neq h(b)$ by the injectivity of f. Similarly, if $a \geq k$ and $b \geq k$, then $h(a) = f(a+1)$, $h(b) = f(b+1)$, so $h(a) \neq h(b)$. Finally, if $a \geq k$ and $b < k$, then $a + 1 > k > b$, so again $h(a) \neq h(b)$. Thus, h is injective. Now, let $p \in A \setminus C$ be arbitrary. We set $n = f^{-1}(p)$. If $n < k$, then $h(n) = p$, and if $n > k$, then $h(n-1) = p$. We cannot have $n = k$, since that would imply that $f(n) = p = f(k) = c \notin A \setminus C$. Thus, f is also surjective, hence bijective.

Hence, $A \setminus C$ has $m - 1$ elements.

We also note the trivial case of $m = 1$, i.e. there exists only one element $x = f(1) \in A$, hence the only subset $C \subseteq A$ having only one element is $C = A = \{x\}$, so $A \setminus C = \emptyset$, which has $m - 1 = 0$ elements.

(ii) We show the result by induction on the number of elements in B. If B is empty, $C \setminus B = C$, and the result is trivial. We establish the base case with B containing 1 element, $x \in B$. Now, if $x \notin C$, $C \setminus B = C$ again. Otherwise, $x \in C$, so $C \setminus B = C \setminus \{x\}$. If this were finite, containing m elements (say), then $(C \setminus \{x\}) \cup \{x\} = C$ would have to contain $m + 1$ elements, and thus be finite as well. This is a contradiction, so $C \setminus B$ is always infinite when B contains exactly 1 element.

Now, suppose that $C \setminus B$ is infinite for all infinite sets C, and for all finite sets B containing exactly n elements. We now choose an arbitrary finite set D containing $n + 1$ elements. Let $x \in D$ be an arbitrary element. Set $F = D \setminus \{x\}$. Thus, F contains n elements (by our previous result). Note that $C \setminus D = (C \setminus F) \setminus \{x\}$. Now, $C \setminus F$ is infinite by our induction hypothesis. Call this set G. We have already shown that $G \setminus \{x\}$ is infinite for any infinite set G. Hence, $C \setminus D$ is infinite for all finite sets D containing $n + 1$ elements.

This completes the proof by induction.

Problem 3. Let $S = \{1, 2\}$ and $T = \{a, b, c\}$.

(i) Determine the number of different injections from S into T.

(ii) Determine the number of differnet surjections from T onto S.

Solution 3.

(i) Let $f\colon S \to T$ be an injection. Each element of S must be have a unique image in T, and each element of T can have at most one pre-image in S. Hence, there are $3 \times 2 = 6$ injections.

(ii) Let $g\colon T \to S$ be a surjection. Each element of S must have at least one pre-image in T. There are $2^3 = 8$ functions from T to S, of which 1 maps all elements to $\{1\}$, and 1 maps all elements to $\{2\}$. Hence, there are $8 - 2 = 6$ surjections.

Problem 4. Exhibit a bijection between $\mathbb{N}$ and the set of all odd integers greater than 13.

Solution 4. We supply the bijection $f\colon \mathbb{N} \to \{15, 17, \dots\}$,

$$f(n) \ = \ 2n + 13, \qquad \text{for all } n \in \mathbb{N}.$$

Clearly, if $p \neq q$, then $f(p) = 2p + 13 \neq 2q + 13 = f(q)$, so f is injective. Also, for arbitrary $m = 2k + 1$, $m > 13$, $k \in \mathbb{N}$, we must have $k > 6$. Thus, we find $n = (m - 13)/2 \in \mathbb{N}$, such that $f(n) = m$. Hence, f is also surjective, and is thus a bijection.

Problem 5. Give an explicit definition of a bijection f from $\mathbb{N}$ onto $\mathbb{Z}$.

Solution 5. We have $f\colon \mathbb{N} \to \mathbb{Z}$,

$$f(n) \ = \ \begin{cases} n/2 & n \text{ is even} \\ -(n-1)/2 & n \text{ is odd} \end{cases}.$$

Problem 6. Exhibit a bijection between $\mathbb{N}$ and a proper subset of itself.

Solution 6. We could reuse the solution from Problem 4. Alternatively, we supply the bijection $g\colon \mathbb{N} \to \mathbb{N} \setminus \{1\}$,

$$f(n) \ = \ n + 1, \qquad \text{for all } n \in \mathbb{N}.$$

Clearly, if $p \neq q$, then $f(p) = p+1 \neq q+1 = f(q)$. Also, for arbitrary $m \in \mathbb{N}\setminus\{1\}$, we find $n = m-1 \in \mathbb{N}$ such that $f(n) = m$. Hence, f is a bijection.

Problem 7. Prove that a set T_1 is denumerable if and only if there is a bijection from T_1 onto a denumerable set T_2.

Solution 7. First, let $f\colon T_1 \to T_2$ be a bijection, where T_2 is denumerable. Then, there exists a bijection $g\colon \mathbb{N} \to T_2$, so the mapping $f^{-1} \circ g\colon \mathbb{N} \to T_2$ is a bijection. Hence, T_1 is also denumerable.

Now, let T_1 be denumerable. Thus, we have a bijection $f\colon \mathbb{N} \to T_1$, whose inverse $f^{-1}\colon T_1 \to \mathbb{N}$ is also a bijection. Since $\mathbb{N}$ is denumerable, we are done.

Problem 8. Give an example of a countable collection of finite sets whose union is not finite.

Solution 8. Let $N_i = \{i\}$, where $i \in \mathbb{N}$. Clearly, each such set is finite, with exactly 1 element. Now, the collection of all such sets, $M = \{N_i : i \in \mathbb{N}\} = \{\{1\}, \{2\}, \dots\}$ is countable. We supply the bijection $f\colon \mathbb{N} \to M$, $f(n) = N_n$, for all $n \in \mathbb{N}$. However, the union of all N_i is simply $\bigcup_{i \in \mathbb{N}} N_i = \{1, 2, \dots\} = \mathbb{N}$, which is infinite.

Problem 9. Prove in detail that if S and T are denumerable, then $S \cup T$ is denumerable.

Solution 9. Let $f\colon \mathbb{N} \to S$ and $g\colon \mathbb{N} \to T$ be bijections. We identify the elements $x_k = f(k)$ and $y_k = g(k)$ for all $n \in \mathbb{N}$, i.e. we write $S = \{x_1, x_2, \dots\}$ and $T = \{y_1, y_2, \dots\}$. Now, we consider two cases.

Case I. $S \cup T = \emptyset$. We construct the bijection $h\colon \mathbb{N} \to S \cup T$,

$$h(n) = \begin{cases} f((n+1)/2) & n \text{ is odd} \\ g(n/2) & n \text{ is even} \end{cases}.$$

Clearly, h is injective since if $a \neq b$, then the injectivity of f and g, together with the disjointedness of S and T means that $h(a) \neq h(b)$. Also, for arbitrary $z \in S \cup T$, we must have exactly one of the following: $z = x_i \in S$, so $h(2i-1) = z$, or $z = y_j \in T$, so $h(2j) = z$. Thus, $S \cup T$ is denumerable.
Case II. $S \cup T \neq \emptyset$. We set $A_1 = A$, $B_1 = B \setminus A$. Then, $A_1 \cap B_1 = \emptyset$, $A_1 \cup B_1 = A \cup B$, and A_1 is denumerable. Now, $B_1 \subseteq B$ is a subset of a countable set B, so is either finite or denumerable. If B_1 is denumerable, then we have A_1, B_1 denumerable, so $A \cup B = A_1 \cup B_1$ is denumerable by Case I. Otherwise, B_1 is finite. Let $B_1 = \{b_1, b_2, \dots, b_m\}$ have m elements. Let $A_1 = \{a_1, a_2, \dots\}$ We construct the bijection $G\colon \mathbb{N} \to A_1 \cup B_1$,

$$F(n) = \begin{cases} b_n & n = 1, 2, \dots m \\ a_{n-m} & n = m+1, m+2, \dots \end{cases}, \qquad \text{for all } n \in \mathbb{N}.$$

Hence, $A \cup B$ is denumerable.

Problem 10.

(i) If (m, n) is the 6th point down the 9th diagonal entry of the array (in the given figure), calculate its number according to the given counting method.

(ii) Given that $h(m, 3) = 19$, find m.

Solution 10.

(i) The first point in the 9th diagonal of the figure is $(1, 9)$. Moving down 6 points, the sum of the coordinates remains contant while the column number increases, so we have $(m, n) = (6, 4)$. Thus, we calculate $h(6, 4) = \frac{1}{2} \cdot 8 \cdot 9 + 6 = 42$.

(ii) Given that $h(m, 3) = \frac{1}{2}(m + 3 - 2)(m + 3 - 1) + m = 19$, we calculate $\frac{1}{2}(m^2 + 3m + 2) + m = \frac{1}{2}(m^2 + 5m + 2) = 19 \implies m^2 + 5m - 36 = 0$. For $m \in \mathbb{N}$, we must have $m = 4$.

Problem 11. Determine the number of elements in $\mathcal{P}(S)$, the collection of all subsets of S, for each of the following sets.

(i) $S = \{1, 2\}$.

(ii) $S = \{1, 2, 3\}$.

(iii) $S = \{1, 2, 3, 4\}$.

Solution 11.

(i) We simply count $\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Hence, there are 4 elements.

(ii) For every element T in the previous power set, we include both T and $T \cup \{3\}$ this time. Hence, we have twice the number of elements in $\mathcal{P}(S)$, i.e. 8 elements.

(iii) Again, we have twice the number of element in $\mathcal{P}(S)$ as before, i.e. 16 elements.

Problem 12. Use mathematical induction to show that if the set S has n elements, then $\mathcal{P}(S)$ has 2^n elements.

Solution 12. We establish the base case with $n = 1$. Let the set $S = \{x_1\}$. Then, $\mathcal{P}(S) = \{\emptyset, \{x_1\}\}$ has exactly $2 = 2^1$ elements.

Now, assume that the given statement holds true for some $k \in \mathbb{N}$. Thus, for any set S with k elements, $\mathcal{P}(S)$ contains 2^k elements. Let T be a finite set of $k+1$ elements, and let $x \in T$ be an element of T. Now, $P = \mathcal{P}(T \setminus \{x\})$ has 2^k elements, since $T \setminus \{x\}$ has k elements. We construct $Q = \{A \cup \{x\} : A \in P\}$, and claim that $P \cap Q = \emptyset$. Indeed, for arbitrary $p \in P$, $x \notin p$ but $x \in q$ for all $q \in Q$. Thus, $P \cup Q$ has $2 \cdot 2^k$ elements, since each element in Q has a one-to-one correspondence with each element in P.

Now, each element $p \in P$ is a subset of T, and so is every element $q \in Q$. Hence, $P \cup Q \subseteq \mathcal{P}(T)$. Also, for any arbitrary element $s \in \mathcal{P}(T)$, we must have one of the following: $x \notin s$, in which case $s \in P$, or $x \in s$, in which case $s \in Q$. Thus, $\mathcal{P}(T) \subseteq P \cup Q$. Hence, $\mathcal{P}(T) = P \cup Q$, and has precisely 2^{k+1} elements. This completes the proof by induction.

Problem 13. Prove that the collection $\mathcal{F}(\mathbb{N})$ of all finite subsets of $\mathbb{N}$ is countable.

Solution 13. We construct the injection $f \colon \mathcal{F}(\mathbb{N}) \to \mathbb{N}$, $f(\emptyset) = 1$,

$$f(S) \ = \ \prod_{n \in S} p_n, \qquad \text{for all } S \in \mathcal{F}(\mathbb{N}) \setminus \{\emptyset\},$$

where p_k is the kth prime number. The uniqueness of the prime factorisation of natural numbers guarantees that f is injective. Thus, $\mathcal{F}(\mathbb{N})$ is countable.

We can do even better with the bijection $g \colon \mathcal{F}(\mathbb{N}) \to \mathbb{N}$, $g(\emptyset) = 1$,

$$g(S) \ = \ 1 + \sum_{n \in S} 2^{n-1}, \qquad \text{for all } S \in \mathcal{F}(\mathbb{N}) \setminus \{\emptyset\}.$$

The bijectivity of g is a consequence of the uniqueness of representation of the natural numbers in binary.

Cantor's Diagonal Argument

We show that the collection of all sequences of natural numbers S, or equivalently $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \dots$, is uncountable.
Assume that S is countable. Note that S must be infinite, because $(n, 0, 0, \dots) \in S$ for all $n \in \mathbb{N}$. Hence, S must be enumerable. Let $f \colon \mathbb{N} \to S$ be a bijection. We thus enumerate every element of S as $s_n = f(n) \in S$ for all $n \in \mathbb{N}$. We notate $s_n(k)$ to be the kth element of the sequence s_n. We construct the sequence, s_0, in the following way: $s_0 \colon \mathbb{N} \to \mathbb{N}$,

$$s_0(k) \ = \ s_k(k) + 1, \qquad \text{for all } k \in \mathbb{N}.$$

Now, s_0 is clearly a sequence of natural numbers, so $s_0 \in S$. Thus, from the bijectivity of f, s_0 has a unique inverse, $f^{-1}(s_0) = c$. This would imply that $s_0 = s_c$, i.e. $s_0(c) = s_c(c)$. However, $s_0(c) = s_c(c) + 1$ by definition. This is a contradiction. Hence, S is uncountable.