MA 1201 : Mathematics II

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Problem 1 Show that the vectors (a, b) and (c, d) in $\mathbb{R}^2$ are linearly independent iff $ad - bc \neq 0$.

Solution We first let (a, b) and (c, d) be linearly indepenedent. Now, if $ad - bc = 0$, then without loss of generality, either $a = b = 0$, in which case (a, b) is the zero vector $(0, 0)$, or $a = 0$ and $d = 0$, in which case one of b or c is also 0, so one of (a, b) and (c, d) is the zero vector $(0, 0)$, or $a = 0$, $d \neq 0$ and $c = 0$, in which case

$$d\,(0, b) \;+\; (-b)\,(0, d) \;=\; (0, 0),$$

or $a, b, c, d \neq 0$, in which case

$$c\,(a, b) \;+\; (-a)\,(c, d) \;=\; (0, 0).$$

This is a contradiction, hence we must have $ad - bc \neq 0$.

Now we assume $ad - bc \neq 0$. If (a, b) and (c, d) were linearly dependent, we find λ, μ, at least one of which is non-zero, such that

$$\lambda(a, b) \;+\; \mu(c, d) \;=\; (0, 0),$$

i.e. $\lambda a = -\mu c$ and $\lambda b = -\mu d$. If $\mu = 0$, that forces $a = b = 0$, and hence $ad - bc = 0$. Similary, $\lambda = 0$ forces $c = d = 0$, and hence $ad - bc = 0$. Finally, for $\lambda, \mu \neq 0$, we have

$$\lambda\mu(ad - bc) \;=\; (\lambda a)(\mu d) - (-\lambda b)(-\mu c) \;=\; 0,$$

and thus $ad - bc = 0$. In all cases, we reach a contradiction. Hence, (a, b) and (c, d) must be linearly independent. ☐

Problem 2 Show that if v and w are linearly independent vectors in $\mathbb{R}^2$, then so are $v + w$ and $v - w$.

Solution We assume the contrary, i.e. we find λ and μ, at least one of which is non-zero, such that

$$\lambda(v + w) + \mu(v - w) \;=\; 0 \implies (\lambda + \mu)v + (\lambda - \mu)w \;=\; \mathbf{0}.$$

The linear independence of v and w demands both $\lambda + \mu = 0$ and $\lambda - \mu = 0$. However, this is only possible if $\lambda = \mu = 0$. This is a contradiction. Hence, $v + w$ and $v - w$ must be linearly independent. ☐

Problem 3 Show that the following are bases of $\mathbb{R}^2$.

(i) $\{(1, 2),\ (4, 3)\}$.

(ii) $\{(1, 1),\ (1, -1)\}$.

Solution

(i) Let $(a, b) \in \mathbb{R}^2$ be arbitrary. We seek $\lambda, \mu \in \mathbb{R}$ such that

$$\lambda(1, 2) \;+\; \mu(4, 3) \;=\; (a, b).$$

Solving the system of equations

$$\begin{aligned} \lambda \;+\; 4\mu \;&=\; a \\ 2\lambda \;+\; 3\mu \;&=\; b \end{aligned}$$

we find $\lambda = (-3a + 4b)/5$ and $\mu = (2a - b)/5$.

We now show that this is a unique solution. Let (λ_1, μ_1) and (λ_2, μ_2) be two pairs of solutions to the above system. If

$$\lambda_1(1,2) + \mu_1(4,3) = \lambda_2(1,2) + \mu_2(4,3) = (a,b),$$

then

$$(\lambda_1 - \lambda_2)(1,2) + (\mu_1 - \mu_2)(4,3) = \mathbf{0}.$$

Using the result in Problem 1, we find that $(1,2)$ and $(4,3)$ are linearly independent, since $1 \cdot 3 - 2 \cdot 4 \neq 0$. Thus, we must have $\lambda_1 - \lambda_2 = \mu_1 - \mu_2 = 0$, thus proving uniqueness.

Hence, any arbitrary vector in $\mathbb{R}^2$ can be uniquely expressed as a linear combination of the two given vectors, i.e. they comprise a basis of $\mathbb{R}^2$. □

(ii) Let $(a,b) \in \mathbb{R}^2$ be arbitrary. Again, we seek $\lambda, \mu \in \mathbb{R}$ such that

$$\lambda(1,1) + \mu(1,-1) = (a,b).$$

Solving the system of equations

$$\begin{aligned} \lambda + \mu &= a \\ \lambda - \mu &= b \end{aligned}$$

we find $\lambda = (a+b)/2$ and $\mu = (a-b)/2$.

We now show that this is a unique solution. Let (λ_1, μ_1) and (λ_2, μ_2) be two pairs of solutions to the above system. If

$$\lambda_1(1,1) + \mu_1(1,-1) = \lambda_2(1,1) + \mu_2(1,-1) = (a,b),$$

then

$$(\lambda_1 - \lambda_2)(1,1) + (\mu_1 - \mu_2)(1,-1) = \mathbf{0}.$$

Using the result in Problem 1, we find that $(1,1)$ and $(1,-1)$ are linearly independent, since $1 \cdot 1 - 1 \cdot (-1) \neq 0$. Thus, we must have $\lambda_1 - \lambda_2 = \mu_1 - \mu_2 = 0$, thus proving uniqueness.

Hence, any arbitrary vector in $\mathbb{R}^2$ can be uniquely expressed as a linear combination of the two given vectors, i.e. they comprise a basis of $\mathbb{R}^2$. □

Problem 4 Show that any set containing 3 vectors in $\mathbb{R}^2$ is linearly dependent. Also show that any linearly independent set of 2 vectors in $\mathbb{R}^2$ is a basis of $\mathbb{R}^2$.

Solution Let $u, v, w \in \mathbb{R}^2$ be arbitrary, with $u = (u_1, u_2)$, $v = (v_1, v_2)$ and $w = (w_1, w_1)$. Note that if any two of u, v, w are linearly dependent, say u and v, then all three are linearly dependent, i.e. if $c_1, c_2 \in \mathbb{R}$, where at least one of them is non-zero, then

$$c_1 u + c_2 v = 0 \implies c_1 u + c_2 v + 0w = \mathbf{0}.$$

Contrapositively, if u, v, w are all linearly indepenedent, then any two of them, say u and v, are also linearly independent.

Assume u, v and w are linearly independent. Then, u and v are linearly independent, i.e. $\Delta_w = u_1 v_2 - u_2 v_1 \neq 0$. Consider the system of equations

$$\begin{aligned} \lambda u_1 + \mu v_1 &= w_1 \\ \lambda u_2 + \mu v_2 &= w_2 \end{aligned}$$

It is easily verified that $\lambda = (w_1 v_2 - w_2 v_1)/\Delta_w$ and $\mu = (u_1 w_2 - u_2 w_1)/\Delta_w$ is a solution to the above system. Moreover, since u and w are linearly independent, $\Delta_v = u_1 w_2 - u_2 w_1 \neq 0$ and since w and v are linearly independent, $\Delta_u = w_1 v_2 - w_2 v_1 \neq 0$. Thus, $\lambda, \mu \neq 0$. However, this means that

$$\lambda u + \mu v - w = \mathbf{0}.$$

This is a contradiction. Hence, any set of 3 vectors in $\mathbb{R}^2$ must be linearly dependent.

Let $u, v \in \mathbb{R}^2$ be linearly independent. We show that they form a basis of $\mathbb{R}^2$. Let $w \in \mathbb{R}^2$ be arbitrary. Like before, we define

$$\begin{aligned} \Delta_w &= u_1 v_2 - u_2 v_1, \\ \Delta_v &= u_1 w_2 - u_2 w_1, \\ \Delta_u &= w_1 v_2 - w_2 v_1. \end{aligned}$$

The linear independence of u and v means that $\Delta_w \neq 0$. Again, it is easily verified that $\lambda u + \mu v = w$, where $\lambda = \Delta_u / \Delta_w$ and $\mu = \Delta_v / \Delta_w$. Furthermore, this solution is unique since if

$$\lambda_1 u + \mu_1 v = \lambda_2 u + \mu_2 v = w,$$

then we must have

$$(\lambda_1 - \lambda_2) u + (\mu_1 - \mu_2) v = \mathbf{0}.$$

The linear independence of u and v demands $\lambda_1 - \lambda_2 = \mu_1 - \mu_2 = 0$.

Hence, an arbitrary vector in $w \in \mathbb{R}^2$ can always be uniquely represented as a linear combination of two linearly independent vectors $u, v \in \mathbb{R}^2$, i.e. $\{u, v\}$ form a basis of $\mathbb{R}^2$.

Problem 5 Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$, $T(x_1, x_2) = (x_1, x_2, 0)$. Show that T is linear and find the matrix of T with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solution Let $x_1, x_2, y_1, y_2, c \in \mathbb{R}$ be arbitrary. We verify

$$T(\mathbf{x}) + T(\mathbf{y}) = (x_1, x_2, 0) + (y_1, y_2, 0) = (x_1 + y_1, x_2 + y_2, 0) = T(x_1 + y_1, x_2 + y_2) = T(\mathbf{x} + \mathbf{y}),$$

$$cT(\mathbf{x}) = c(x_1, x_2, 0) = (cx_1, cx_2, 0) = T(cx_1, cx_2) = T(c\mathbf{x}).$$

Hence, T is linear. Let $V = \{\mathbf{v_1}, \mathbf{v_2}\}$ and $W = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ be the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Then,

$$[T]_V^W = ([T(\mathbf{v_1})]_W \quad [T(\mathbf{v_2})]_W) = (T(1,0) \quad T(0,1)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 6 Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T(x_1, x_2) = (2x_1, -3x_2)$. Show that T is linear and find the matrix of T with respect to the standard basis of $\mathbb{R}^2$.

Solution Let $x_1, x_2, y_1, y_2, c \in \mathbb{R}$ be arbitrary. We verify

$$T(\mathbf{x}) + T(\mathbf{y}) = (2x_1, -3x_2) + (2y_1, -3y_2) = (2x_1 + 2y_1, -3x_2 - 3y_2) = T(x_1 + y_1, x_2 + y_2) = T(\mathbf{x} + \mathbf{y}),$$

$$cT(\mathbf{x}) = c(2x_1, -3x_2) = (2cx_1, -3cx_2, 0) = T(cx_1, cx_2) = T(c\mathbf{x}).$$

Hence, T is linear. Clearly,

$$[T] = (T(1,0) \quad T(0,1)) = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}.$$

Problem 7 Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$, $T(x, y) = (x, x+y, y)$. Find the matrix of T with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solution Let $V = \{\mathbf{v_1}, \mathbf{v_2}\}$ and $W = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ be the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Then,

$$T(\mathbf{v_1}) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad T(\mathbf{v_2}) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Thus,

$$[T] = (T(\mathbf{v_1}) \quad T(\mathbf{v_2})) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Problem 8

(i) Show that $\{(2,1,1),(1,2,2),(1,1,1)\}$ is linearly dependent in $\mathbb{R}^3$.

(ii) Show that $\{(1,2,2),(2,1,2),(2,2,1)\}$ is linearly independent in $\mathbb{R}^3$.

(iii) Show that $\{(0,1,1),(1,0,1),(1,1,0)\}$ is a basis of $\mathbb{R}^3$.

Solution

(i) We supply the relation

$$\begin{pmatrix}2\\1\\1\end{pmatrix} + \begin{pmatrix}1\\2\\2\end{pmatrix} - 3\begin{pmatrix}1\\1\\1\end{pmatrix} \;=\; \mathbf{0}.$$

(ii) Let $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\begin{pmatrix}1\\2\\2\end{pmatrix} + c_2\begin{pmatrix}2\\1\\2\end{pmatrix} + c_3\begin{pmatrix}2\\2\\1\end{pmatrix} \;=\; \mathbf{0}.$$

We obtain the system of linear equations

$$\begin{aligned} c_1 + 2c_2 + 2c_3 &= 0, & (1)\\ 2c_1 + c_2 + 2c_3 &= 0, & (2)\\ 2c_1 + 2c_2 + c_3 &= 0. & (3)\end{aligned}$$

Now, (1) + (2) - $\frac{3}{2}$(3) gives $\frac{5}{2}c_3 \;=\; 0$. Hence, from (1) and (2), $c_1 = -2c_2 = -2(-2c_1)$, from which we have $c_1 = c_2 = c_3 = 0$. Moreover, Cramer's rule tells us that this solution is unique, since the determinant

$$\begin{vmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{vmatrix} \;=\; (1+8+8)-(4+4+4) \;=\; 5 \neq 0.$$

This proves that the given set of vectors are linearly independent.

(iii) Let $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\begin{pmatrix}0\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\\1\end{pmatrix} + c_3\begin{pmatrix}1\\1\\0\end{pmatrix} \;=\; \mathbf{0}.$$

We obtain the system of linear equations

$$\begin{aligned} c_2 + c_3 &= 0, & (1)\\ c_1 + c_3 &= 0, & (2)\\ c_1 + c_2 &= 0. & (3)\end{aligned}$$

Now, (1) + (2) - (3) gives $2c_3 \;=\; 0$. Hence, from (1) and (2), $c_1 = c_2 = c_3 = 0$. Moreover, Cramer's rule tells us that this solution is unique, since the determinant

$$\begin{vmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{vmatrix} \;=\; (0+1+1)-(0+0+0) \;=\; 2 \neq 0.$$

This proves that the given set of vectors are linearly independent.

Let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ be arbitrary. We seek $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$a_1\begin{pmatrix}0\\1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\0\\1\end{pmatrix} + a_3\begin{pmatrix}1\\1\\0\end{pmatrix} \;=\; v.$$

Like before, this is equivalent to solving the system of linear equations

$$\begin{aligned} a_2 + a_3 &= v_1, \\ a_1 + a_3 &= v_2, \\ a_1 + a_2 &= v_3, \end{aligned}$$

whose solution exists and is unique from Cramer's rule. It is easily verified that

$$\begin{aligned} 2a_1 &= -v_1 + v_2 + v_3, \\ 2a_2 &= v_1 - v_2 + v_3, \\ 2a_3 &= v_1 + v_2 - v_3. \end{aligned}$$

Hence, any vector $v \in \mathbb{R}^3$ is uniquely expressible in terms of the given vectors, which proves that they are a basis of $\mathbb{R}^3$.

Problem 9 Let $T, S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (x, 0)$ and $S(x, y) = (0, y)$, where $x, y \in \mathbb{R}$. Find the mappings $S \circ T$ and $T \circ S$. Let $[L]$ be the matrix representation of a linear mapping L in the standard basis. Check that $[S \circ T] = [S]\,[T]$, with respect to the standard basis of $\mathbb{R}^2$.

Solution Let $x, y \in \mathbb{R}$ be arbitrary. We have $(S \circ T), (T \circ T)\colon \mathbb{R}^2 \to \mathbb{R}^2$,

$$\begin{aligned} (S \circ T)(x, y) &= S(T(x, y)) = S(x, 0) = (0, 0), \\ (T \circ S)(x, y) &= T(S(x, y)) = T(0, y) = (0, 0). \end{aligned}$$

In the standard basis of $\mathbb{R}^2$,

$$\begin{aligned} T(1, 0) &= (1, 0), & T(0, 1) &= (0, 0), \\ S(1, 0) &= (0, 0), & S(0, 1) &= (0, 1), \\ (S \circ T)(1, 0) &= (0, 0), & (S \circ T)(0, 1) &= (0, 0). \end{aligned}$$

Hence,

$$[T] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad [S] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad [S \circ T] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easily verified that

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $[S \circ T] = [S]\,[T]$.

Problem 10 Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, such that $T(x, y, z) = (3x - 2y + z, x - 3y - 2z)$, for $x, y, z \in \mathbb{R}^3$. Find $[T]$ with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.

Solution We calculate

$$T(1, 0, 0) = (3, 1), \qquad T(0, 1, 0) = (-2, -3), \qquad T(0, 0, 1) = (1, -2).$$

Hence, we have

$$[T] = \begin{pmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{pmatrix}.$$

Problem 11 Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, and let $\beta = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ and $\gamma = \{(1, 0), (1, 1)\}$ be bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Let

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$$

be the matrix representation of T in the bases β and γ. Find $T(x, y, z)$.

Solution We have

$$\begin{aligned}[T(0,1,1)]_\gamma &= (1,2) & [T(1,0,1)]_\gamma &= (2,1) & [T(1,1,0)]_\gamma &= (4,0)\\ T(0,1,1) &= (1,0)+2(1,1) & T(1,0,1) &= 2(1,0)+(1,1) & T(1,1,0) &= 4(1,0)\\ T(0,1,1) &= (3,2) & T(1,0,1) &= (3,1) & T(1,1,0) &= (4,0)\end{aligned}$$

We reuse the calculation in Problem 8(iii) to note that the coordiantes of an arbitrary vector $v = (x, y, z) \in \mathbb{R}^3$, in the basis β are given by $[v]_\beta = ((y+z-x)/2, (x+z-y)/2, (x+y-z)/2)$. Hence,

$$T(x,y,z) = \frac{1}{2}(y+z-x)\,(3,2)+\frac{1}{2}(x+z-y)\,(3,1)+\frac{1}{2}(x+y-z)\,(4,0) = \left(2x+2y+z, \frac{1}{2}(-x+y+3z)\right).$$