MA 1201 : Mathematics II

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Solution 1. Let $\epsilon > 0$. Since g is Riemann integrable on $[a, b]$, we find $\delta_0 \in \mathbb{R}$ such that for all tagged partitions $\dot{P}$ of $[a, b]$ such that $\|P\| \leq \delta_0$, we have

$$|S(g, \dot{P}) - \int_a^b g| < \frac{\epsilon}{2}.$$

Let $\dot{Q}$ be a tagged partition on $[a, b]$. Note that since $f(x) - g(x) = 0$ everywhere except at $x = c$, and c is a tag of at most 2 intervals,

$$S(f, \dot{Q}) - S(g, \dot{Q}) \ \leq \ 2|f(c) - g(c)|\|\dot{Q}\|.$$

Thus, setting $\delta = \min\{\delta_0,\ \epsilon/(4|f(c) - g(c)| + 4)\}$, for all partitions such that $\|\dot{P}\| \leq \delta$, we have

$$\begin{aligned}
|S(f, \dot{P}) - \int_a^b g| \ &= \ |S(f, \dot{P}) - S(g, \dot{P}) + S(g, \dot{P}) - \int_a^b g| \\
&\leq \ |S(f, \dot{P}) - S(g, \dot{P})| + |S(g, \dot{P}) - \int_a^b g| \\
&\leq \ \frac{|f(c) - g(c)|}{|f(c) - g(c)| + 1} \cdot \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&< \ \epsilon.
\end{aligned}$$

Hence, f is Riemann integrable on $[a, b]$, and

$$\int_a^b f \ = \ \int_a^b g.$$

□

Solution 2. Let $\epsilon > 0$. We seek $k \in \mathbb{N}$ such that for all $n \geq k$, $n \in \mathbb{N}$,

$$|S(f, \dot{P}_n) - \int_a^b f| < \epsilon.$$

Since f is Riemann integrable, there exists $\delta \in \mathbb{R}$ such that for all partitions $\dot{P}$ such that $\|\dot{P}\| < \delta$,

$$|S(f, \dot{P}) - \int_a^b f| < \epsilon.$$

Note that since $\|\dot{P}_n\| \to 0$ as $n \to \infty$, there exists $k' \in \mathbb{N}$ such that for all $n \geq k'$, $\|\dot{P}_n\| < \delta$.
Hence, setting $k = k'$ finishes the proof.

$$\int_a^b f \ = \ \lim_{n \to \infty} S(f, \dot{P}_n).$$

□

Solution 3. Let $f\colon [0, 1] \to \mathbb{R}$ be defined such that $f(x) = \frac{1}{2n}$ for all $x = \frac{1}{n}$, $n \in \mathbb{N}$ and $f(x) = 0$ otherwise. We claim that f is Riemann integrable, and that $\int_0^1 f = 0$.
Let $\epsilon > 0$. We seek δ such that for all tagged partitions $\dot{P}$ on $[0, 1]$ such that $\|\dot{P}\| < \delta$, we have $|S(f, \dot{P})| < \epsilon$.

We set $E = \{x : x \in [0, 1] \land f(x) \geq \epsilon/2\}$. This set is finite.

Given a partition $\dot{P}$, a point $x \in E$ can be a tag of at most two intervals in $\dot{P}$. Also, $f(x) \le \frac{1}{2}$ for each of these points. The total length of each interval is at most $\|\dot{P}\|$, and there are k such intervals. Hence, the contribution to the Riemann sum over those intervals containing such points is at most $\frac{1}{2} \cdot 2k \cdot \|\dot{P}\|$. In the remaining intervals, each tag $z \in [0,1] \setminus E$, so $f(z) < \epsilon/2$. The total length of these intervals is at most the length of the domain, i.e. 1. Hence, their contribution to the Riemann sum is at most $\epsilon/2 \cdot 1$.

We set $\delta = \epsilon/2k$.[1] Then, for all partitions such that $\|\dot{P}\| < \delta$,

$$\begin{aligned} S(f,\dot{P}) &= \sum_{\xi_i \in E} f(\xi_i)(x_{i+1}-x_i) + \sum_{\xi_i \notin E} f(\xi_i)(x_{i+1}-x_i) \\ &< \sum_{\xi_i \in E} \frac{1}{2} \cdot \frac{\epsilon}{2k} + \sum_{\xi_i \notin E} \frac{\epsilon}{2}(x_{i+1}-x_i) \\ &\le \frac{1}{2} \cdot \frac{\epsilon}{2k} \cdot 2k + \frac{\epsilon}{2} \cdot 1 \\ &= \epsilon. \end{aligned}$$

This completes the proof. □

Solution 4.

(i)

$$\lim_{n\to\infty} \sum_{k=1}^{3n} \frac{1}{n+k} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{3n} \frac{1}{1+k/n} = \int_0^3 \frac{1}{1+x}\,\mathrm{d}x = \log 4.$$

(ii)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sin\frac{k\pi}{n} = \int_0^1 \sin(\pi x)\,\mathrm{d}x = 2.$$

(iii)

$$\lim_{n\to\infty} \sum_{k=1}^{2n} \frac{n}{n^2+k^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{2n} \frac{1}{1+k^2/n^2} = \int_0^2 \frac{1}{1+x^2}\,\mathrm{d}x = \arctan 2.$$

(iv)

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{k}{n}\right)^{1/n} = \exp \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log\left(1+\frac{k}{n}\right) = \exp \int_0^1 \log(1+x)\,\mathrm{d}x = 4e^{-1}.$$

(v)

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{k^2}{n^2}\right)^{k/n^2} = \exp \lim_{n\to\infty} \frac{1}{n} \prod_{k=1}^{n} \frac{k}{n} \log\left(1+\frac{k^2}{n^2}\right) = \exp \int_0^1 x\log(1+x^2)\,\mathrm{d}x = 2e^{-1/2}.$$

Solution 5.

(i) We claim that if $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable, then f is bounded.

Suppose not. Let the Riemann integral of f on $[a,b]$ be L. Then, for $\epsilon = 1$, we find δ such that for all tagged partitions $\dot{P}$ on $[a,b]$ with $\|\dot{P}\| < \delta$, we have $|S(f,\dot{P}) - L| < 1$, i.e. $S(f,\dot{P}) < |L| + 1$.

Let $Q = \{x_0, x_1, \dots, x_n\}$ be such a partition, with $\|Q\| < \delta$. Since f is unbounded on $[a,b]$, it must be unbounded on at least one of the subintervals $[x_k, x_{k+1}]$. Now, we select tags to create the tagged partition $\dot{Q} = \{([x_i, x_{i+1}], \xi_i)\}$. We choose $\xi_k \in [x_k, x_{k+1}]$ such that

$$|f(\xi_k)(x_{k+1}-x_k)| > |L| + 1 + |\sum_{i\neq k} f(\xi_i)(x_{i+1}-x_i)|.$$

[1] In the case where $k = 0$, i.e. $\epsilon > 1$, the result follows trivially since $f(x) < 1$ for all $x \in [0,1]$.

Thus,

$$|S(f,\dot{Q})| \;\geq\; |f(\xi_k)(x_{k+1}-x_k)| - |\sum_{i\neq k} f(\xi_i)(x_{i+1}-x_i)| \;>\; |L|+1.$$

This is a contradiction, which proves our claim. $\square$

(ii) For any tagged partition $\dot{P}$ on $[a,b]$,

$$S(f,\dot{P}) \;\leq\; \sum_i |f(\xi_i)|(x_{i+1}-x_i) \;\leq\; M(b-a).$$

Hence, for all $\epsilon > 0$, there exists δ such that for all such partitions with $\|\dot{P}\| < \delta$,

$$|\,|S(f,\dot{P})| - |\int_a^b f|\,| \;\leq\; |S(f,\dot{P}) - \int_a^b f| \;<\; \epsilon$$

$$\left|\int_a^b f\right| \;<\; |S(f,\dot{P})| + \epsilon \;<\; M(b-a)+\epsilon.$$

Since this holds for all $\epsilon > 0$, we can write

$$\left|\int_a^b f\right| \;\leq\; M(b-a).$$

$\square$

Solution 6.

(i) We have $f\colon [-2,2] \to \mathbb{R}$,

$$f(x) = \begin{cases} 3x^2\cos\dfrac{\pi}{x^2} + 2\pi\sin\dfrac{\pi}{x^2} & x \in [-2,2]\setminus\{0\}, \\ 0 & x = 0. \end{cases}$$

We set $F\colon [-2,2] \to \mathbb{R}$,

$$F(x) = \begin{cases} x^3\cos\dfrac{\pi}{x^2} & x \in [-2,2]\setminus\{0\}, \\ 0 & x = 0. \end{cases}$$

Now, f is continuous on $[-2,2]\setminus\{0\}$, and hence is Riemann integrable. Also, F is continuous on $[-2,2]$, and $F'(x) = f(x)$ for all $x \in [-2,2]\setminus\{0\}$. Using the Fundamental Theorem of Calculus,

$$\int_{-2}^{+2} f \;=\; F(2) - F(-2) \;=\; 16\cos\frac{\pi}{4}.$$

(ii) We have $f\colon [0,3] \to \mathbb{R}$,

$$f(x) = \begin{cases} -x & x \in [0,1], \\ x & x \in (1,3]. \end{cases}$$

We set $F\colon [0,3] \to \mathbb{R}$,

$$F(x) = \begin{cases} \frac{-x^2}{2} & x \in [0,1], \\ \frac{x^2}{2} - 1 & x \in (1,3]. \end{cases}$$

$$\int_0^3 f \;=\; F(3) - F(0) \;=\; \frac{7}{2}.$$

(iii) We have $f\colon [1,3] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & x \in [1,2), \\ 2 & x \in [2,3), \\ 3 & x = 3 \end{cases}$$

We set $F\colon [1,3] \to \mathbb{R}$,

$$F(x) = \begin{cases} x & x \in [1,2), \\ 2x-2 & x \in [2,3), \\ 3x-5 & x = 3. \end{cases}$$

$$\int_1^3 f \;=\; F(3) - F(1) \;=\; 3.$$

Solution 7. We have $f\colon [0,3] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & x \in [0,1), \\ x & x \in [1,2), \\ 2x & x \in [2,3), \\ 3x & x = 3 \end{cases}$$

We set $F\colon [0,3] \to \mathbb{R}$,

$$F(x) = \begin{cases} 0 & x \in [0,1), \\ \frac{x^2}{2} - \frac{1}{2} & x \in [1,2), \\ \frac{2x^2}{2} - \frac{5}{2} & x \in [2,3), \\ \frac{3x^2}{2} - \frac{14}{2} & x = 3. \end{cases}$$

$$\int_0^3 f \;=\; F(3) - F(0) \;=\; \frac{13}{2}.$$