MA 1201 : Mathematics II

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Solution 1.

(i) The sum $\sum_{n=1}^{\infty} \frac{n}{5n+11}$ diverges, since as $n \to \infty$, $\frac{n}{5n+11} \to \frac{1}{5} \neq 0$.

(ii) Note that the series $\sum_{n=0}^{\infty} r^n$ converges when $0 < r < 1$. Furthermore, the sum of this series is $\frac{1}{1-r}$. Hence, the corresponding sums for $r = \frac{3}{5}$ and $r = \frac{4}{5}$ are $\frac{5}{2}$ and 5 respectively. Thus the sum of these two series must converge to $\frac{15}{2}$.

(iii) Note that

$$\frac{3^n + 5^n}{4^n} > 1 > 0,$$

and the series $\sum_{n=0}^{\infty} 1$ clearly diverges. Hence, the series $\sum_{n=0}^{\infty} \frac{3^n+5^n}{4^n}$ diverges by the comparison test.

(iv) The sum $\sum_{n=1}^{\infty} \sin(n\pi/2)$ diverges, since the limit as $n \to \infty$ of $\sin(n\pi/2)$ does not exist.

(v) We calculate the partial sums

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{k^2 + 5k + 6} \\ &= \sum_{k=1}^{n} \frac{1}{k+2} - \frac{1}{k+3} \\ &= \frac{1}{3} - \frac{1}{n+2}. \end{aligned}$$

Clearly, the sequence of partial sums $\{S_n\}_n$ converges, since as $n \to \infty$, $\frac{1}{n+2} \to 0$ and $S_n \to \frac{1}{3}$. Hence, the sum of the series is $\frac{1}{3}$.

(vi) We calculate the partial sums

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{k^2 + 2k} \\ &= \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+2} \\ &= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right). \end{aligned}$$

As $n \to \infty$, $S_n \to \frac{3}{4}$, which is the sum of the series.

(vii) We calculate the partial sums

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} \\ &= \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} \\ &= \frac{1}{2} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) - \left(\frac{1}{k+1} - \frac{1}{k+2} \right) \\ &= \frac{1}{2} \left(1 - \frac{1}{n+1} - \frac{1}{2} + \frac{1}{n+2} \right). \end{aligned}$$

As $n \to \infty$, $S_n \to \frac{1}{4}$, which is the sum of the series.

(viii) For $\sum_{n=1}^{\infty} \cos n$ to converge, we must have $\cos n \to 0$ as $n \to \infty$. This would imply that $\cos(n+1) \to 0$, which means $\cos n \cos 1 - \sin n \sin 1 \to 0$. This requires $\sin n \to 0$. However, $\cos^2 n + \sin^2 n = 1$. Thus, taking the limit on the left yields 0, a contradiction. Hence, the series diverges.

Solution 2. Let $\{X_n\}_n$, $\{Y_n\}_n$ and $\{Z_n\}_n$ be the sequences of partial sums of the series $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ and $\sum_{n=1}^{\infty}(x_n + y_n)$ respectively.
We seek series such that $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty}(x_n + y_n)$ converge. Thus, as $n \to \infty$, the sequences of partial sums X_n and Z_n must both converge. Now,

$$Z_n - X_n \;=\; \sum_{k=1}^{n}(x_n + y_n) - \sum_{k=1}^{n} x_n \;=\; \sum_{k=1}^{n} y_n \;=\; Y_n.$$

Thus, the difference of these convergent sequences of partial sums, which is Y_n, must converge. However, this means that the series $\sum_{n=1}^{\infty} y_n$ must also converge.
Hence, it is impossible to choose x_n and y_n as demanded.

Solution 3.

(i) Note that $n^3 - 5n + 7 = n(n^2 - 5) + 7 > 0$ for all $n \in \mathbb{N}$. We take the limit

$$\lim_{n\to\infty} \frac{\frac{n+8}{n^3-5n+7}}{\frac{1}{n^2}} \;=\; \lim_{n\to\infty} \frac{n^3 + 8n^2}{n^3 - 5n + 7} \;=\; 1 \neq 0.$$

As the series $\sum_{n=1}^{\infty} 1/n^2$ converges, the given series must also converge.

(ii) Note that $n(n+6)^2 = n^3 + 12n^2 + 36n > n^3 + 2$ for all $n \in \mathbb{N}$. Thus,

$$0 \leq \frac{1}{\sqrt{n}} \leq \frac{n+6}{\sqrt{n^3+2}}.$$

As the series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges, the given series must also diverge.

(iii) For $n > 20$, each term of the given series is positive. We take the limit

$$\lim_{n\to\infty} \frac{\sqrt{5n} - 10}{3n + \sqrt{n}} \cdot \frac{\sqrt{n}}{1} \;=\; \lim_{n\to\infty} \frac{\sqrt{5}n - 10\sqrt{n}}{3n + \sqrt{n}} \;=\; \frac{\sqrt{5}}{3} \neq 0.$$

As the series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges, the given series must also diverge.

(iv) We use the inequality $\log x < x$ for all $x > 0$. Setting $x = \sqrt{n}$, we have $\log n < 2\sqrt{n}$ for all $n \in \mathbb{N}$. Thus,

$$0 \leq \frac{\log n}{n^2} \leq \frac{2}{n^{3/2}}.$$

As the series $\sum_{n=1}^{\infty} 1/n^{3/2}$ converges, the given series must also converge.

(v) Note that

$$0 \leq \sqrt[3]{n^3+1} - n \;=\; \frac{1}{\sqrt[3]{(n^3+1)^2} + n\sqrt[3]{n^3+1} + n^2} \;<\; \frac{1}{\sqrt[3]{(n^3)^2} + n\sqrt[3]{n^3} + n^2} \;=\; \frac{1}{3n^2}.$$

As the series $\sum_{n=1}^{\infty} 1/n^2$ converges, the given series must also converge.

(vi) Note that

$$0 \leq \frac{1}{1+2^n} < \frac{1}{2^n}.$$

As the series $\sum_{n=1}^{\infty}(1/2)^n$ converges, the given series must also converge.

(vii) Note that as $n \to \infty$, $2^{-n} \to 0$. Hence, the given series diverges as $1/(1 + 2^{-n}) \to 1$.

(viii) We use the inequality $\sin x \geq 2x/\pi$, for all $x \in [0, \pi/2]$. Setting $x = \pi/2n$, we have

$$0 \leq \frac{1}{n} \leq \sin\frac{\pi}{2n}$$

for all $n \in \mathbb{N}$. As the series $\sum_{n=1}^{\infty} 1/n$ diverges, the given series must also diverge.

(ix) Note that for all $n \in \mathbb{N}$,

$$0 \leq \frac{1}{4n} \leq \frac{n}{(2n-1)(2n+1)}.$$

As the series $\sum_{n=1}^{\infty} 1/n$ diverges, the given series must also diverge.

(x) Note that for all $n \in \mathbb{N}$,

$$\frac{1}{n^{p-1}} \leq \frac{n+1}{n^p} \leq \frac{2}{n^{p-1}}.$$

This means that the given series converges precisely when the series $\sum_{n=1}^{\infty} 1/n^{p-1}$ converges, i.e. when $p > 2$. Otherwise, it diverges.

Solution 4. Since $a_n, b_n > 0$ for all $n \in \mathbb{N}$, we use the AM-GM inequality to write

$$0 \leq a_n b_n \leq \frac{1}{2}(a_n^2 + b_n^2).$$

Note that the series $\sum_{n=1}^{\infty}(a_n^2 + b_n^2)/2$ converges, since it is a linear combination of two convergent series $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$. Hence, the series $\sum_{n=1}^{\infty} a_n b_n$ also converges. □

Solution 5. From $\lim_{n\to\infty} a_n/b_n = +\infty$, we find $k \in \mathbb{N}$ such that for all $n \geq k$, $n \in \mathbb{N}$, $a_n/b_n > G = 1 > 0$. Thus, for all $n \geq k$, we have $0 \leq b_n \leq a_n$.
Hence, the series $\sum_{n=1}^{\infty} a_n$ diverges if the series $\sum_{n=1}^{\infty} b_n$ diverges. □

Solution 5. Since $a_n > 0$, we have

$$b_n \; = \; \frac{1}{n}\sum_{k=1}^{n} a_k \geq \frac{1}{n} \cdot a_1 > 0.$$

As the series $\sum_{n=1}^{\infty} 1/n$ diverges, the given series must also diverge. □