# MA 1101 : Mathematics I

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**Solution 1.**
Let $a < b$ and let $f\colon [a, b] \to \mathbb{R}$ be convex. We claim that

$$\max\{f(a), f(b)\} \geq f(x), \text{ for all } x \in (a, b).$$

To prove this, let $x \in (a, b)$ be given. We set $M = \max\{f(a), f(b)\}$ and $\lambda = (b - x)/(b - a)$. Clearly, $\lambda > 0$ and $1 - \lambda = (x - a)/(b - a) > 0$, thus $\lambda \in [0, 1]$.
By the convexity of $f$, we have

$$\begin{aligned} f(\lambda a + (1 - \lambda)b) &\leq \lambda f(a) + (1 - \lambda)f(b) \\ f(x) &\leq \lambda f(a) + (1 - \lambda)f(b) \\ &\leq \lambda M + (1 - \lambda)M \\ &= M \end{aligned}$$

This proves the desired statement. □

**Solution 2.**
Let $a < b$ and let $f\colon (a, b) \to \mathbb{R}$ be differentiable. We claim that $f$ is convex if and only if

$$f(y) - f(x) \geq f'(x)(y - x), \text{ for all } x, y \in (a, b) \qquad (\star)$$

To prove this, we first assume that $f$ is convex. We will show that $(\star)$ holds.
If $x = y$, the result follows trivially. Let $y > x$. We choose $\alpha \in (a, b)$ such that $y > x > \alpha > a$. Using the Rising Slope Theorem, we have

$$\frac{f(y) - f(x)}{y - x} \geq \frac{f(x) - f(\alpha)}{x - \alpha}.$$

Taking the limit as $\alpha \to x$, we have

$$\frac{f(y) - f(x)}{y - x} \geq f'(x).$$

Again, if $x > y$, we choose $\beta \in (a, b)$ such that $b > \beta > x > y$. Using the Rising Slope Theorem, we have

$$\frac{f(\beta) - f(x)}{\beta - x} \geq \frac{f(x) - f(y)}{x - y}.$$

Taking the limit as $\beta \to x$, we have

$$f'(x) \geq \frac{f(x) - f(y)}{x - y}.$$

In either case,

$$f(y) - f(x) \geq f'(x)(y - x), \text{ for all } x, y \in (a, b).$$

We now assume that $(\star)$ holds. We will show that $f$ is convex.
Let $x, y, z \in (a, b)$, such that $x > y > z$. Using $(\star)$, we have

$$f(x) - f(y) \geq f'(y)(x - y)$$

$$f(z) - f(y) \geq f'(y)(z - y)$$

Rearranging,

$$\frac{f(x) - f(y)}{x - y} \geq f'(y)$$

$$\frac{f(z)-f(y)}{z-y} \le f'(y)$$

This is the same as

$$\frac{f(x)-f(y)}{x-y} \ge \frac{f(y)-f(z)}{y-z}.$$

Therefore, using the Rising Slope Theorem, $f$ is convex.
This proves the desired result. $\square$

**Solution 3.**
Let $n \in \mathbb{N}$, let $a_i, \lambda_i > 0$ for all $i = 1, \ldots, n$, and let $p \ge 1$. We claim that

$$\frac{\sum \lambda_i a_i^p}{\sum \lambda_i} \ge \left(\frac{\sum \lambda_i a_i}{\sum \lambda_i}\right)^p.$$

To prove this, we define $f\colon (0,\infty) \to \mathbb{R}$ by $f(x) := x^p$ for all $x \in (0,\infty)$. Note that $f''(x) = p(p-1)x^p \ge 0$ for all $x \in (0,\infty)$. Hence, $f$ is convex.
Using Jensen's Inequality on $a_i$, with weights $\lambda_i / \sum \lambda_i$, we have

$$f\left(\frac{\sum \lambda_i a_i}{\sum \lambda_i}\right) \le \frac{\sum \lambda_i f(a_i)}{\sum \lambda_i},$$

from which the desired statement follows directly. $\square$

**Solution 4.**

(i) Let $a > 0$. We claim that for all $x \ge y > 0$,

$$\frac{a^x - 1}{x} \ge \frac{a^y - 1}{y}.$$

To prove this, note that when $x = y$, the inequality follows trivially. Assume $x > y$.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := a^x$ for all $x \in \mathbb{R}$. Note that $f''(x) = a^x(\log a)^2 \ge 0$ for all $x \in \mathbb{R}$. Hence, $f$ is convex.

We set $\lambda = y/x$. Note that $\lambda > 0$ and $1 - \lambda = (x-y)/x > 0$. Thus, $\lambda \in [0,1]$.

By the convexity of $f$, we have

$$\begin{aligned}
f(\lambda x + (1-\lambda)0) &\le \lambda f(x) + (1-\lambda) f(0) \\
f(y) &\le \frac{y}{x} f(x) + \frac{(x-y)}{x} f(0) \\
x a^y &\le y a^x + (x-y)a^0 \\
x a^y - x &\le y a^x - y \\
\frac{a^y - 1}{y} &\le \frac{a^x - 1}{x}
\end{aligned}$$

This proves the desired statement. $\square$

(ii) We claim that for all $n \in \mathbb{N}$,

$$\left(1 + \frac{1}{n+1}\right)^{n+1} \ge \left(1 + \frac{1}{n}\right)^n.$$

To prove this, we apply Bernoulli's inequality on the following.

$$
\begin{aligned}
\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} &= \left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1}\left(1+\frac{1}{n}\right) \\
&= \left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1}\left(1+\frac{1}{n}\right) \\
&= \left(1-\frac{1}{(n+1)^2}\right)^{n+1}\left(1+\frac{1}{n}\right) \\
&\geq \left(1-\frac{n+1}{(n+1)^2}\right)\left(1+\frac{1}{n}\right) \\
&= \frac{n}{n+1}\cdot\frac{n+1}{n} \\
&= 1
\end{aligned}
$$

$\square$