MA 1101 : Mathematics I

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Solution 1. Let $\emptyset \neq D \subseteq \mathbb{R}$, let $c \in D$ and let $f\colon D \to \mathbb{R}$ be continuous at c with $f(c) > 0$. We claim that there exists $\delta > 0$ such that

$$f(x) > 0, \text{ for all } x \in (c-\delta, c+\delta) \cap D.$$

Since f is continuous at c, we find $\delta_c > 0$ such that

$$|f(x) - f(c)| < \frac{1}{2} f(c), \text{ for all } x \in (c-\delta_c, c+\delta_c) \cap D.$$

Suppose that our claim is false, i.e. there exists at least one $x_0 \in (c-\delta_c, c+\delta_c) \cap D$ such that $f(x_0) \leq 0$. Then, $f(x_0) - f(c) < 0 \Rightarrow |f(x_0) - f(c)| = f(c) - f(x_0) \geq f(c)$, a contradiction. Hence, setting $\delta = \delta_c$ proves our claim. □

Solution 2. Let $\emptyset \neq D \subseteq \mathbb{R}$, let $c \in D$ and let $f, g\colon D \to R$ be continuous at c.

(i) We claim that $f + g$ is continuous at c.
Let $\epsilon > 0$ be given. We find δ_f, δ_g such that for all $x \in D$,

$$|x - c| < \delta_f \implies |f(x) - f(c)| < \epsilon/2,$$

$$|x - c| < \delta_g \implies |g(x) - g(c)| < \epsilon/2.$$

We set $\delta = \min\{\delta_f, \delta_g\}$. Then, for all $x \in D$ satisfying $|x - c| < \delta$, we have

$$\begin{aligned} |(f(x) + g(x)) - (f(c) + g(c))| &= |(f(x) - f(c)) + (g(x) - g(c))| \\ &\leq |f(x) - f(c)| + |g(x) - g(c)| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

This proves our claim. □

(ii) We claim that for all $\alpha \in \mathbb{R}$, αf is continuous at c.
Let $\epsilon > 0$ be given. If $\alpha \neq 0$, we find δ_f such that for all $x \in D$,

$$|x - c| < \delta_f \implies |f(x) - f(c)| < \epsilon/|\alpha|.$$

We set $\delta = \delta_f$. Then, for all $x \in D$ satisfying $|x - c| < \delta$, we have

$$\begin{aligned} |\alpha f(x) - \alpha f(c)| &= |\alpha||f(x) - f(c)| \\ &< |\alpha| \frac{\epsilon}{|\alpha|} \\ &= \epsilon \end{aligned}$$

If $\alpha = 0$, we trivially have

$$|x - c| < \delta = \epsilon \implies |\alpha f(x) - \alpha f(c)| = 0 < \epsilon.$$

This proves our claim. □

(iii) We claim that fg is continuous at c.

Let $\epsilon > 0$ be given. We find $\delta_1, \delta_2, \delta_3, \delta_4$ such that for all $x \in D$,

$$\begin{aligned} |x-c| < \delta_1 &\implies |f(x) - f(c)| < \sqrt{\epsilon/2}, \\ |x-c| < \delta_2 &\implies |g(x) - g(c)| < \sqrt{\epsilon/2}, \\ |x-c| < \delta_3 &\implies |f(x) - f(c)| < \epsilon/4(1+|g(c)|), \\ |x-c| < \delta_4 &\implies |g(x) - g(c)| < \epsilon/4(1+|f(c)|). \end{aligned}$$

We set $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$. Then, for all $x \in D$ satisfying $|x-c| < \delta$, we have

$$\begin{aligned} |(fg)(x) - (fg)(c)| &= |f(x)g(x) - f(c)g(c)| \\ &= |(f(x) - f(c) + f(c))(g(x) - g(c) + g(c)) - f(c)g(c)| \\ &= |(f(x) - f(c))(g(x) - g(c)) + f(c)(g(x) - g(c)) + g(c)(f(x) - f(c)) + f(c)g(c) - f(c)g(c)| \\ &= |(f(x) - f(c))(g(x) - g(c)) + f(c)(g(x) - g(c)) + g(c)(f(x) - f(c))| \\ &\leq |f(x) - f(c)||g(x) - g(c)| + |f(c)||g(x) - g(c)| + |g(c)||f(x) - f(c)| \\ &< \sqrt{\frac{\epsilon}{2}}\sqrt{\frac{\epsilon}{2}} + \frac{|f(c)|\epsilon}{4(1+|f(c|)} + \frac{|g(c)|\epsilon}{4(1+|g(c|)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \epsilon \end{aligned}$$

This proves our claim. $\square$

(iv) We claim that if $g(c) \neq 0$, f/g is continuous at c. To prove this, we first show that $h\colon D \to \mathbb{R}$, $h(x) = 1/g(x)$ is continuous at c.

Let $\epsilon > 0$ be given. We find δ_1, δ_2 such that for all $x \in D$,

$$\begin{aligned} |x-c| < \delta_1 &\implies |g(x) - g(c)| < \frac{1}{2}|g(c)|, \\ |x-c| < \delta_2 &\implies |g(x) - g(c)| < \frac{1}{2}\epsilon|g(c)|^2. \end{aligned}$$

We set $\delta = \min\{\delta_1, \delta_2\}$. Then, for all $x \in D$ satisfying $|x-c| < \delta$, we have

$$\begin{aligned} \frac{1}{2}|g(c)| &> |g(x) - g(c)| \\ &\geq ||g(x)| - |g(c)|| \\ &\geq |g(c)| - |g(x)| \\ |g(x)| &> \frac{1}{2}|g(c)| > 0 \\ \frac{1}{|g(x)|} &< \frac{2}{|g(c)|} \\ |h(x) - h(c)| &= \left|\frac{1}{g(x)} - \frac{1}{g(c)}\right| \\ &= \frac{|g(x) - g(c)|}{|g(c)g(x)|} \\ &= |g(x) - g(c)|\frac{1}{|g(c)||g(x)|} \\ &< \frac{1}{2}\epsilon|g(c)|^2\frac{2}{|g(c)|^2} \\ &= \epsilon \end{aligned}$$

Thus, h is continuous at c. Therefore, $f/g = fh$ is continuous at c. $\square$

Solution 3. Let $I \subseteq \mathbb{R}$ be an open interval, let $c \in I$ and let $f, g \colon D \to R$ be differentiable at c. Note that f, g are continuous at c.
Since f, g are differentiable at c, we have the following.

$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ g'(c) &= \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \end{aligned}$$

(i) We claim that $f + g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$.
Note that

$$\begin{aligned} f'(c) + g'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c} \\ &= \lim_{x \to c} \frac{(f(x) + g(x)) - (f(c) + g(c))}{x - c} \\ &= \lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x - c} \\ &= (f+g)'(c) \end{aligned}$$

Hence,

$$(f+g)'(c) = \lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x - c} = f'(c) + g'(c)$$

This proves our claim. □

(ii) We claim that for all $\alpha \in \mathbb{R}$, αf is differentiable at c and $(\alpha f)'(c) = \alpha f'(c)$.
Note that

$$\begin{aligned} \alpha f'(c) &= \alpha \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c} \frac{\alpha f(x) - \alpha f(c)}{x - c} \\ &= \lim_{x \to c} \frac{(\alpha f)(x) - (\alpha f)(c)}{x - c} \\ &= (\alpha f)'(c) \end{aligned}$$

Hence,

$$(\alpha f)'(c) = \lim_{x \to c} \frac{(\alpha f)(x) - (\alpha f)(c)}{x - c} = \alpha f'(c)$$

This proves our claim. □

(iii) We claim that fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.
Note that since c is a limit point of I, $f(c) = \lim_{x \to c} f(x)$ and $g(c) = \lim_{x \to c} g(x)$.

$$\begin{aligned} f'(c)g(c) + f(c)g'(c) &= g(c) \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} f(x) \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= \lim_{x \to c} \frac{(f(x) - f(c))g(c) + f(x)(g(x) - g(c))}{x - c} \\ &= \lim_{x \to c} \frac{f(x)g(c) - f(c)g(c) + f(x)g(x) - f(x)g(c)}{x - c} \\ &= \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x - c} \\ &= \lim_{x \to c} \frac{(fg)(x) - (fg)(c)}{x - c} \\ &= (fg)'(c) \end{aligned}$$

Hence,

$$(fg)'(c) \;=\; \lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c} \;=\; f'(c)g(c)+f(c)g'(c)$$

This proves our claim. $\square$

(iv) We claim that if $g(c)\neq 0$, f/g is differentiable at c and $(f/g)'(c) = (f'(c)g(c) - f(c)g'(c))/g(c)^2$. To prove this, we first show that $h\colon D\to\mathbb{R}$, $h(x) = 1/g(x)$ is differentiable at c and $h'(c) = -g'(c)/g(c)^2$.

Note that h is continuous and c is a limit point of I, hence $h(c) = \lim_{x\to c} h(x)$.

$$\begin{aligned}
-\frac{g'(c)}{g(c)^2} \;&=\; -\frac{1}{g(c)^2}\lim_{x\to c}\frac{g(x)-g(c)}{x-c}\\
&=\; \frac{1}{g(c)}\lim_{x\to c}\frac{1}{g(x)}\lim_{x\to c}\frac{g(c)-g(x)}{x-c}\\
&=\; \lim_{x\to c}\frac{\frac{1}{g(x)}-\frac{1}{g(c)}}{x-c}\\
&=\; \lim_{x\to c}\frac{h(x)-h(c)}{x-c}\\
&=\; h'(c)
\end{aligned}$$

Hence,

$$h'(c) \;=\; -g'(c)/g(c)^2$$

Using the product rule,

$$(f/g)'(c) \;=\; (fh)'(c) \;=\; f'(c)h(c)+f(c)h'(c) \;=\; f'(c)/g(c) - f(c)g'(c)/g(c)^2$$

This proves our claim. $\square$

Solution 4.

(i) We claim that for all $x>0$,

$$\frac{x}{1+x} < \ln(1+x) < x.$$

Let $f, g\colon (0,\infty)\to\mathbb{R}$ be defined as follows.

$$f(x) \;=\; \ln(1+x) - \frac{x}{1+x},\ \text{for all } x>0,$$

$$g(x) \;=\; x - \ln(1+x),\ \text{for all } x>0,$$

We note that

$$\begin{aligned}
f'(x) \;&=\; \frac{1}{1+x} - \frac{(1+x)-x}{(1+x)^2}\\
&=\; \frac{(1+x)-(1+x)+x}{(1+x)^2}\\
&=\; \frac{x}{(1+x)^2}\\
&>\; 0\\
g'(x) \;&=\; 1-\frac{1}{1+x}\\
&=\; \frac{(1+x)-1}{1+x}\\
&=\; \frac{x}{1+x}\\
&>\; 0
\end{aligned}$$

Thus, f and g are monotonically increasing on $(0, \infty)$. We can write

$$f(x) > \lim_{t \to 0} f(t) = 0$$
$$g(x) > \lim_{t \to 0} g(t) = 0$$

Therefore,

$$\ln(1+x) > \frac{x}{1+x}$$
$$x > \ln(1+x)$$

This proves our claim. $\square$

(ii) We claim that for all $x > 0$,

$$e^x > 1 + x + \frac{1}{2}x^2.$$

Let $f \colon [0, x] \to \mathbb{R}$ be defined as $f(t) = e^t$, for all $t \in [0, x]$. Clearly, f is continuous in $[0, x]$ and differentiable in $(0, x)$. Note that $f'(t) = f(t) = e^t$. Hence, f, f' are continuous on $[0, x]$ and $f'' = f$ exists in $(0, x)$.

Using Taylor's Theorem, we find $c \in (0, x)$ such that

$$e^x = e^0 + e^0(x - 0) + \frac{1}{2}e^c(x - 0)^2.$$

Since, $e^0 = 1$ and $e^c > 1$ for $c > 0$, we have

$$e^x > 1 + x + \frac{1}{2}x^2.$$

This proves our claim. $\square$

(iii) We claim that for all $x, y \in \mathbb{R}$,

$$|\sin x - \sin y| \leq |x - y|.$$

Note that if $x = y$, our claim is trivially true.

Without loss of generality, let $x > y$. Let $f, g \colon [x, y] \to \mathbb{R}$ be defined as follows.

$$f(t) = \sin t, \text{ for all } t \in [x, y],$$

$$g(t) = t, \text{ for all } t \in [x, y].$$

Clearly, f and g are continuous in $[x, y]$ and differentiable in (x, y). Note that $f'(t) = \cos t$ and $g'(t) = 1$.

Using Cauchy's Mean Value Theorem, we find $c \in (x, y)$ such that.

$$(\sin x - \sin y) = (x - y) \cos c.$$

Since $\cos c \leq 1$,

$$|\sin x - \sin y| \leq |x - y|.$$

This proves our claim. $\square$