MA 1101 : Mathematics I

Problem 1. (Maximum of convex functions)
Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be convex. Prove that

$$\max\{f(a), f(b)\} \geqslant f(x), \text{ for all } x \in (a, b).$$

In other words, a convex function on $[a, b]$ attains its global maximum at one of the points a and b.

Problem 2.
Let $a < b$ and let $f : (a, b) \to \mathbb{R}$ be differentiable. Prove that f is convex if and only if

$$f(y) - f(x) \geqslant f'(x)(y - x), \text{ for all } x, y \in (a, b).$$

Problem 3. (Weighted p-th power inequality)
Let $n \in \mathbb{N}$, let $a_i, \lambda_i > 0$ for all $i = 1, \ldots, n$, and let $p \geqslant 1$. Prove that,

$$\frac{\sum_{i=1}^{n} \lambda_i a_i^p}{\sum_{i=1}^{n} \lambda_i} \geqslant \left(\frac{\sum_{i=1}^{n} \lambda_i a_i}{\sum_{i=1}^{n} \lambda_i} \right)^p.$$

Problem 4.
Establish the following inequalities.

(i) Let $a > 0$. Then, for all $x \geqslant y > 0$,

$$\frac{a^x - 1}{x} \geqslant \frac{a^y - 1}{y}.$$

(ii) For all $n \in \mathbb{N}$,

$$\left(1 + \frac{1}{n+1}\right)^{n+1} \geqslant \left(1 + \frac{1}{n+1}\right)^{n+1}.$$