MA 1101 : Mathematics I

Problem 1.
Find limit points of $S \subseteq \mathbb{R}$, where

(i) S is a finite set.

(ii) $S := (0, \infty)$.

(iii) $S := [1, 2) \cup \{3\}$.

(iv) $S := [1, 2) \cup (2, 3)$.

(v) $S := \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

(vi) $S := \left\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\right\}$.

Problem 2.
Use the (ϵ, δ)-definition to prove the existence or non-existence of the following limits.

(i) $f : \mathbb{R} \to \mathbb{R}$, $f(x) := [x]$, $\lim_{x\to 0} f(x)$.

(ii) $f : \mathbb{R} \to \mathbb{R}$, $f(x) := [x] - \left[\frac{x}{3}\right]$, $\lim_{x\to 0} f(x)$.

(iii) $f : \mathbb{R} \setminus \{2\} \to \mathbb{R}$, $f(x) := \frac{x^3-8}{x-2}$, $\lim_{x\to 2} f(x)$.

(iv) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) := x \sin \frac{1}{x}$, $\lim_{x\to 0} f(x)$.

(v) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) := \frac{x}{|x|}$, $\lim_{x\to 0} f(x)$.

Problem 3.
Let $\emptyset \neq D \subset \mathbb{R}$, let $f : D \to \mathbb{R}$ and let $a \in \mathbb{R}$ be a limit point of D. Let us suppose that $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist and we write

$$L := \lim_{x\to a} f(x),\ M := \lim_{x\to a} g(x).$$

Show that

(i) $\lim_{x\to a}(f(x) + g(x)) = L + M$.

(ii) For all $\alpha \in \mathbb{R}$, $\lim_{x\to a}(\alpha f(x)) = \alpha L$.

(iii) $\lim_{x\to a} f(x)g(x) = LM$.

(iv) If $M \neq 0$,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$