MA 1101 : Mathematics I

Problem 1. (Ring of integers)
Define an equivalence relation $\sim_{\mathbb{Z}}$ on $\mathbb{N} \times \mathbb{N}$ as

$$(m,n) \sim_{\mathbb{Z}} (p,q) \text{ if } m+q = n+p.$$

Let us set

$$\mathbb{Z} := (\mathbb{N} \times \mathbb{N})/\sim_{\mathbb{Z}},$$
$$\bar{0} := [(1,1)],\ \bar{1} := [(2,1)],\ \mathbb{Z}^+ := \{[(n+1,1)] : n \in \mathbb{N}\}.$$

For $a = [(m,n)]$, $b = [(p,q)] \in \mathbb{Z}$, we define

$$a + b := [(m+p, n+q)],\ a \cdot b := [(mp+nq, mq+np)].$$

Prove that

(i) **Addition :**
  (a) $+$ is well-defined, associative and commutative.
  (b) $a + \bar{0} = a = \bar{0} + a$, for all $a \in \mathbb{Z}$.
  (c) For all $a \in \mathbb{Z}$, there exists a unique $x \in \mathbb{Z}$ satisfying $a + x = \bar{0} = x + a$. We denote x as $-a$ and say that $-a$ is the *negative* of a.
  (d) For all $a, b \in \mathbb{Z}$, there exists a unique $x \in \mathbb{Z}$ satisfying $a + x = b$.

(ii) **Multiplication :**
  (a) $\cdot$ is well-defined, associative and commutative.
  (b) $a \cdot \bar{1} = a = \bar{1} \cdot a$, for all $a \in \mathbb{Z}$.

(iii) **Distributivity :** For all $a, b, c \in \mathbb{Z}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

(iv) **No zero divisors :** For all $a, b \in \mathbb{Z}$ with $a, b \neq \bar{0}$, we have $a \cdot b \neq \bar{0}$.

(v) **Cancellation :** For all $a, b, c \in \mathbb{Z}$ with $a \neq \bar{0}$, we have $a \cdot b = a \cdot c \Rightarrow b = c$.

(vi) **Order** : For all $a, b \in \mathbb{Z}$, we say that $a > b$ if $a - b := a + (-b) \in \mathbb{Z}^+$. Show that, for all $a, b \in \mathbb{Z}$, we have $a \cdot b > 0$ if $a, b > 0$ or $a, b < 0$.

(vii) **Identification map :** Define $I_{\mathbb{N}} : \mathbb{N} \to \mathbb{Z}$ by

$$I_{\mathbb{N}}(n) := [(n+1, 1)],\ \text{for all } n \in \mathbb{N}.$$

Show that
  (a) $I_{\mathbb{N}}$ is one-one.
  (b) $I_{\mathbb{N}}(\mathbb{N}) = \mathbb{Z}^+$.
  (c) $I_N(1) = \bar{1}$.
  (d) $I_{\mathbb{N}}(m+n) = I_{\mathbb{N}}(m) + I_N(n)$, for all $m, n \in \mathbb{N}$.
  (e) $I_{\mathbb{N}}(m \cdot n) = I_{\mathbb{N}}(m) \cdot I_{\mathbb{N}}(n)$, for all $m, n \in \mathbb{N}$.
  (f) $I_{\mathbb{N}}(m) > I_{\mathbb{N}}(n)$, for all $m, n \in \mathbb{N}$ with $m > n$.

Problem 2. (Field of rationals)
Define an equivalence relation $\sim_{\mathbb{Q}}$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ as

$$(m,n) \sim_{\mathbb{Q}} (p,q) \text{ if } mq = np.$$

Let us set

$$\mathbb{Z} := (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})) / \sim_{\mathbb{Q}},$$
$$\bar{0} := [(0,1)],\ \bar{1} := [(1,1)].$$

For $a = [(m,n)]$, $b = [(p,q)] \in \mathbb{Q}$, we define

$$a + b := [(mq+np, nq)],\ a \cdot b := [(mp, nq)].$$

Prove that

(i) **Addition :**

(a) $+$ is well-defined, associative and commutative.

(b) $a + \bar{0} = a = \bar{0} + a$, for all $a \in \mathbb{Q}$.

(c) For all $a \in \mathbb{Q}$, there exists a unique $x \in \mathbb{Q}$ satisfying $a + x = \bar{0} = x + a$. We denote x as $-a$ and say that $-a$ is the *negative* of a.

(d) For all $a, b \in \mathbb{Q}$, there exists a unique $x \in \mathbb{Q}$ satisfying $a + x = b$.

(ii) **Multiplication :**

(a) $\cdot$ is well-defined, associative and commutative.

(b) $a \cdot \bar{1} = a = \bar{1} \cdot a$, for all $a \in \mathbb{Q}$.

(c) For all $a \in \mathbb{Q} \setminus \{\bar{0}\}$, there exists a unique $x \in \mathbb{Q}$ satisfying $a \cdot x = \bar{1} = x \cdot a$. We denote x as a^{-1} and say that a^{-1} is the *inverse* of a.

(d) For all $a, b \in \mathbb{Q} \setminus \{\bar{0}\}$, there exists a unique $x \in \mathbb{Q}$ satisfying $a \cdot x = b$.

(iii) **Distributivity :** For all $a, b, c \in \mathbb{Q}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

(iv) **No zero divisors :** For all $a, b \in \mathbb{Q}$ with $a, b \neq \bar{0}$, we have $a \cdot b \neq \bar{0}$.

(v) **Cancellation :** For all $a, b, c \in \mathbb{Q}$ with $a \neq \bar{0}$, we have $a \cdot b = a \cdot c \Rightarrow b = c$.

(vi) **Order** : For all $a, b \in \mathbb{Q}$, we say that $a > b$ if $mq > np$ where $a = [(m, n)]$, $b = [(p, q)]$ with $n, q \in \mathbb{N}$. Show that, for all $a, b \in \mathbb{Q}$, we have $a \cdot b > 0$ if $a, b > 0$ or $a, b < 0$.

(vii) **Identification map :** Define $I_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Q}$ by

$$I_{\mathbb{Z}}(n) := [(n, 1)], \text{ for all } n \in \mathbb{Z}.$$

Show that

(a) $I_{\mathbb{Z}}$ is one-one.

(b) $I_{\mathbb{Z}}(0) = \bar{0}$, $I_{\mathbb{Z}}(1) = \bar{1}$.

(c) $I_{\mathbb{Z}}(m + n) = I_{\mathbb{Z}}(m) + I_{Z}(n)$, for all $m, n \in \mathbb{Z}$.

(d) $I_{\mathbb{Z}}(m \cdot n) = I_{\mathbb{Z}}(m) \cdot I_{\mathbb{Z}}(n)$, for all $m, n \in \mathbb{Z}$.

(e) $I_{\mathbb{Z}}(m) > I_{\mathbb{Z}}(n)$, for all $m, n \in \mathbb{Z}$ with $m > n$.