MA 1101 : Mathematics I

Problem 1.
For all $n \in \mathbb{N}$,

(i) $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$.

(ii) $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.

(iii) $1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{1}{3}\left(4n^3 - n\right)$.

(iv) $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$.

(v) $\sum_{r=1}^{n} r(r+1)\cdots(r+9) = \frac{1}{11}n(n+1)\cdots(n+10)$.

Problem 2.
Prove that

(i) $3^n > n^2$, for all $n \in \mathbb{N}$.

(ii) $(1+x)^n \geqslant 1 + nx$, for all $x > -1$ and $n \in \mathbb{N}$.

(iii) $\binom{2n}{n} < 2^{2n-2}$, for all $n \geqslant 5$, $n \in \mathbb{N}$.

Problem 3.
Prove that

(i) Every $n \in \mathbb{N}$, $n \geqslant 2$, has a prime divisor/factor.

(ii) **Fibonacci sequence :** We define the sequence $(f_n)_{n \geqslant 0}$ as

$$f_0 = 0, f_1 = 1, \text{ and } f_n := f_{n-1} + f_{n-2}, \text{ for all } n \geqslant 2.$$

Prove that, for all $n \in \mathbb{N}$,

(a) $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$.

(b) $f_1 + \cdots + f_{2n-1} = f_{2n}$.

(c) $f_2 + \cdots + f_{2n} = f_{2n+1} - 1$.