MA 1101 : Mathematics I

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We wish to solve the cubic equation with real coefficients

$$ax^3 + 3bx^2 + 3cx + d = 0$$

1 Depressed cubic

Substituting $x = y - \frac{b}{a}$ yields the cubic

$$y^3 + 3qy + r = 0$$

where $q = (ac - b^2)/a^2$ and $r = (2b^3 - 3abc + a^2d)/a^3$.

2 Cardano's method

We set $y = u + v$. Cubing, we have

$$y^3 - 3uvy - (u^3 + v^3) = 0$$

Comparing coefficients with our depressed cubic, we have the system of equations

$$\begin{cases} u^3 + v^3 & = -r \\ u^3v^3 & = -q^3 \end{cases}$$

Thus, u^3 and v^3 are simply roots of the quadratic

$$t^2 + rt - q^3 = 0$$

We set

$$u^3 = \frac{-r + \sqrt{r^2 + 4q^3}}{2}$$
$$v^3 = \frac{-r - \sqrt{r^2 + 4q^3}}{2}$$

Taking cube roots and selecting appropriate u, v which satisfy the original system yields the desired roots of our cubic.

3 Identities

Let the roots of our depressed cubic be α, β, γ. By Vieta's formula, we have

$$\begin{cases} \alpha + \beta + \gamma & = 0 \\ \alpha\beta + \beta\gamma + \gamma\alpha & = 3q \\ \alpha\beta\gamma & = -r \end{cases}$$

With this, we deduce the following identities.

$$\begin{aligned}
\sum \alpha^2 &= \left(\sum \alpha\right)^2 - 2\sum \alpha\beta \\
&= -6q
\end{aligned}$$

$$\begin{aligned}
\sum \alpha^3 &= \left(\sum \alpha\right)^3 - 3(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) \\
&= -3r
\end{aligned}$$

$$\begin{aligned}
\sum \alpha^2\beta^2 &= \left(\sum \alpha\beta\right)^2 - 2\sum \alpha^2\beta\gamma \\
&= (3q)^2 - 2\alpha\beta\gamma \sum \alpha \\
&= 9q^2
\end{aligned}$$

$$\begin{aligned}
\sum \alpha^3\beta^3 &= \left(\sum \alpha\beta\right)^3 - 3\prod(\alpha\beta+\beta\gamma) \\
&= (3q)^3 - 3\alpha\beta\gamma(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) \\
&= 27q^3 + 3r^2
\end{aligned}$$

$$\begin{aligned}
\sum \alpha^2\beta + \sum \alpha\beta^2 &= \sum \alpha\beta(\alpha+\beta) \\
&= -\sum \alpha\beta\gamma \\
&= 3r
\end{aligned}$$

$$\begin{aligned}
\left(\sum \alpha^2\beta\right)\left(\sum \alpha\beta^2\right) &= \sum \alpha^3\beta^3 + \sum \alpha^2\beta(\beta\gamma^2 + \gamma\alpha^2) \\
&= \sum \alpha^3\beta^3 + \sum \alpha^2\beta^2\gamma^2 + \sum \alpha^4\beta\gamma \\
&= \sum \alpha^3\beta^3 + 3\alpha^2\beta^2\gamma^2 + \alpha\beta\gamma\sum \alpha^3 \\
&= 27q^3 + 3r^2 + 3r^2 + 3r^2 \\
&= 27q^3 + 9r^2
\end{aligned}$$

$$\begin{aligned}
(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2 &= \left(\sum \alpha\beta^2 - \sum \alpha^2\beta\right)^2 \\
&= \left(\sum \alpha\beta^2 + \sum \alpha^2\beta\right)^2 - 4\left(\sum \alpha\beta^2\right)\left(\sum \alpha^2\beta\right) \\
&= (3r)^2 - 4(27q^3 + 9r^2) \\
&= -27(4q^3 + r^2)
\end{aligned}$$

$$\begin{aligned}
\sum(\alpha-\beta)(\beta-\gamma) &= \sum(\alpha\beta - \alpha\gamma - \beta^2 + \beta\gamma) \\
&= \sum \alpha\beta - \sum \alpha\gamma - \sum \beta^2 + \sum \beta\gamma \\
&= -\sum \alpha^2 + \sum \alpha\beta \\
&= 6q + 3q \\
&= 9q
\end{aligned}$$

$$
\begin{aligned}
\sum \alpha^4 &= \left(\sum \alpha^2\right)^2 - 2\sum \alpha^2\beta^2 \\
&= (-6q)^2 - 2(9q^2) \\
&= 18q^2
\end{aligned}
$$

$$
\begin{aligned}
\sum \alpha^3\beta - \sum \alpha\beta^3 &= \sum \alpha\beta(\alpha^2 - \beta^2) \\
&= \sum \alpha\beta(-\gamma)(\alpha - \beta) \\
&= -\alpha\beta\gamma \sum (\alpha - \beta) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\sum \alpha^3\beta \ = \ \sum \alpha\beta^3 &= \frac{1}{2}\left(\sum \alpha^3\beta + \sum \alpha\beta^3\right) \\
&= \frac{1}{2}\left(\sum \alpha^3\beta + \sum \alpha^3\gamma\right) \\
&= \frac{1}{2}\sum \alpha^3(\beta + \gamma) \\
&= -\frac{1}{2}\sum \alpha^4 \\
&= -9q^2
\end{aligned}
$$

4 Cubic discriminant

We set

$$
\Delta \ = \ a^4(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = -27a^4(4q^3 + r^2)
$$

Now,

$$
\begin{aligned}
a^6(4q^3 + r^2) &= 4(ac - b^2)^3 + (2b^3 - 3abc + a^2d)^2 \\
&= 4(ac - b^2)(ac - b^2)^2 + (2b(b^2 - ac) - a(bc - ad))^2 \\
&= -4(b^2 - ac)(b^2 - ac)^2 + 4b^2(b^2 - ac)^2 - 4ab(b^2 - ac)(bc - ad) + a^2(bc - ad)^2 \\
&= 4(b^2 - ac)(-(b^2 - ac)^2 + b^2(b^2 - ac) - ab(bc - ad)) + a^2(bc - ad)^2 \\
&= 4(b^2 - ac)(-b^4 + 2ab^2c - a^2c^2 + b^4 - ab^2c - ab^2c + a^2bd) + a^2(bc - ad)^2 \\
&= 4(b^2 - ac)(-a^2c^2 + a^2bd) + a^2(bc - ad)^2 \\
&= -4a^2(b^2 - ac)(c^2 - bd) + a^2(bc - ad)^2
\end{aligned}
$$

Thus,

$$
\frac{\Delta}{27} \ = \ 4(b^2 - ac)(c^2 - bd) - (bc - ad)^2 = -a^4(4q^3 + r^2)
$$

Clearly, if any two roots of our cubic are equal, we have $\Delta = 0$.
Conversely, if $\Delta = 0$, our cubic must have a repeated root. Furthermore, this repeated root has to be real, since if it were complex, it's complex conjugate must also be a root, yielding 3 complex roots to our cubic, which is absurd. Hence, all 3 roots of our cubic must be real.